

# The classical Schrödinger equation\*

K. R. W. Jones

*School of Physics, University of Melbourne,  
Parkville 3052, Melbourne, Australia.*

(Dated: Original Preprint: 10 December 1991)

## Abstract

Using a simple geometrical construction based upon the linear action of the Heisenberg–Weyl group we deduce a new nonlinear Schrödinger equation that provides an exact dynamic and energetic model of any classical system whatsoever, be it integrable, nonintegrable or chaotic. Within our model classical phase space points are represented by equivalence classes of wavefunctions that have identical position and momentum expectation values. Transport of these equivalence classes is effected in a manner that avoids dispersion and thereby leads to a system of wavefunction dynamics such that the expectation values track classical trajectories *precisely* for arbitrarily long times. Interestingly, the value of  $\hbar$  proves immaterial for the purpose of constructing this alternative version of classical mechanics. The new feature which  $\hbar$  does mediate concerns a surprising embedding of Berry’s phase within ordinary classical mechanics. Some interesting problems are exposed concerning inclusion of the projection postulate within this model nonlinear system and we discover a remarkable route for the recovery of the ordinary linear theory.

PACS numbers: 0230, 0365, 0545

---

\* Author’s note: This is the original text of a pre-print that was lost for some twenty years. Over the years, people have asked me for a copy but I had none. This work is *not refereed* and was never published (since it was lost). Given what I know now, I would write it differently, but place it here under Creative Commons 3.0 - Attribution License under the condition that this footnote is displayed. It is old, but may be of interest for the ideas it explores. Opinions expressed may have been superseded by later works.

## I. INTRODUCTION

This paper describes a rather intuitive mathematical result that is suggestive within itself of some rather deep speculations of a physical nature. At issue shall be the construction of a novel nonlinear integrodifferential wave equation which is dynamically and energetically equivalent to Hamiltonian classical mechanics, but which must differ from it in its attendant epistemology. It should therefore be understood at the outset that this paper lies upon the fringes of current orthodoxy. Nevertheless, we believe it should prove to be of the utmost utility in refining our understanding of that orthodoxy. With these factors in mind we first outline an heuristic pathway via which we were led to consider a specific mathematical question that is of intrinsic physical interest. From that point of departure our argument is able to proceed in a precise and logical fashion.

The deepest puzzle of non-relativistic quantum theory must surely be that, whereas classical dynamics may in general be nonlinear, the underlying, and supposedly fundamental, theory of quantum mechanics employs purely linear dynamics. An observation of this kind has motivated a recurrent historical interest in the problem of how one might successfully introduce nonlinearity into the Schrödinger equation[1, 2]. Perhaps the most persistent reason for that is the idea that a nonlinearity of the right variety might well suppress wavepacket dispersion and thereby permit the existence of soliton solutions having properties closer to the familiar concept of a classical particle[2]. A related rationale concerns the current irreconcilable duality of quantum dynamics vis a vis the two distinct processes of unitary evolution and wavepacket collapse[3]. In this connection the idea has been advanced that the stochastic process of collapse might be subsumed as a possible nonlinear evolutionary process that is somehow characteristic of quantum measurements[4, 5].

Whatever the rationale, the most common approach to the introduction of quantal nonlinearity has been to consider the addition of mild perturbative terms to the standard Schrödinger equation. Perhaps the best known paper of this variety is that by Biälynicki-Birula and Mycielski[2]. Their study presented a particular kind of logarithmic nonlinearity as being appropriate to the physical constraint that separated systems should combine in a manner that avoids their quantal entanglement in the absence of a mutual interaction. That work led to some careful empirical tests via neutron interferometry which placed quite stringent upper bounds upon the strength of the logarithmic quantal nonlinearity[6, 7].

Recently Weinberg[8, 9] has revived the topic via the elucidation of a far more general theory of nonlinear quantal evolution. This theory is distinguished by its use of the projective character of the quantum space of states to extend the proper treatment of separated systems beyond the class of purely logarithmic nonlinearities. Although the stated motivation remains that of providing theoretical guidance to the precision tests of quantal linearity, where the implicit expectation is that of a null result, the theory itself is very elegant. So much so that one might believe in it were it not for the fact that we have yet to observe quantal nonlinearity of any kind in any system.

Given that the implications of the discovery of quantal nonlinearity would be so profound, and noting that Weinberg's recent contribution actually is a theory of quite general character, it is our belief that the current approach to the topic of experiment is perhaps too negative. What one may need, if nonlinearity is true physics, is some guidance about a new place to look where there has not previously been an incentive to look. In this respect the close formal similarity between Weinberg's theory and Hamiltonian classical mechanics suggests that the discussion of quantal nonlinearity might now fruitfully be widened to include the larger question of the connection between classical and quantum mechanics, and the curious possibility that nonlinearity may actually be connected with the classical limit. In this respect it is significant that two notable precursors[10, 11] of Weinberg's formalism, were guided by aspects of the formal similarity between classical and quantal dynamics rather than an investigation into possible quantal nonlinearities.

An alternative motivation for studying nonlinear quantum mechanics concerns the apparent absence[12] of dynamical chaos within the ordinary linear theory of quantum mechanics. In stating this position we are influenced considerably by the recent paper of Ford et al[13]. Given that dynamical chaos is present within the nonlinear regime of the special theory of classical mechanics, it becomes plausible to suppose, for the sake of exploration, that the border regions of the classical limit may well be just that special place where quantal nonlinearity could become a strong rather than a weak effect. Such an idea is heretical, but cannot be immediately discounted. The reason why lies in the fact that an experimental probing of systems that are neither manifestly quantal nor manifestly classical presents peculiar problems of both definition and design within the current orthodoxy. One prefers to have a clean system of definite type. Even to imagine otherwise begs of a theoretical guide we do not as yet have, namely a prototype mesoquantal mechanics.

Returning now to the presence or absence of quantal nonlinearity, it is a fact that all null tests performed to date[14] have been carried out upon systems which *were already known* to be well described by linear quantum theory. There exists therefore the slim but enticing possibility that modern physics may have let something slip through its net. In the exploration of this question one therefore desires a guiding principle that would help locate a candidate mesoquantal theory. The obvious necessity is that a theory of this kind must subsume both ordinary linear quantum theory and ordinary classical mechanics. As such we should expect it to remain a theory of wavefunctions. Note that linear quantum theory already achieves this aim in large part by showing that classical mechanics is a good approximation in the limit  $\hbar \rightarrow 0$ . We must therefore add to the above a reason for believing that the nature of things might be more subtle. In this respect the apparent reality of classical dynamical chaos and the general absence of macroscopic superpositions provide some minimal guidance that one might legitimately pursue a nonlinear theory.

These few tenuous clues, coupled with the formal similarity between Weinberg's generalized nonlinear evolution theory and classical mechanics, motivated the author to look specifically at the theory of classical mechanics as one possible manifestation of a nonlinear quantum theory. Our approach to this exercise was to pose the following simple question:

*Can one find an evolution equation for wavefunctions in Hilbert space such that the expectation values of quantal position and momentum operators will precisely follow the trajectories of an arbitrarily chosen classical system?*

Having posed this question we were somewhat surprised to find an affirmative answer. The route we shall take leads to a general and seemingly very natural nonlinear quantum evolution equation. The motivation of that equation arises from the desire to construct a quantal model of classical mechanics. However, its possible physical role transcends that special application.

Although Weinberg's theory acted primarily as a spur to our interest, it is considered remarkable that the dynamical structure to which we are led seems to share a close affinity with his. To the extent that Weinberg's theory resembles Heisenberg's mechanics, our contribution resembles its natural Schrödinger analogue. Indeed, at the end of the paper we shall demonstrate how knowledge of our own result enabled the author to find a precisely analogous classical correspondence result within Weinberg's theory. Elsewhere, we reported

that argument as stemming from a plausible ansatz[15]; here one may discern its deeper origin.

In summary, the goal of this paper shall be the deduction of a new nonlinear wave equation which models exact classical mechanics as the non-dispersive evolution of wavefunctions in a manner that proves to be independent of  $\hbar$ . Subsequent papers shall present further elaborations of this result in relation to quantization theory, the classical limit and the possibility of mesoquantal physics.

## II. OUTLINE

At heart our work depends upon a simple group theoretic result. We shall merely replace the ordinary kinematical group of motions upon the phase space of classical mechanics, the Abelian group of additive translations, by its projective cousin, the Heisenberg–Weyl group[16]. Everything else follows from this single step. However, in order to make the argument widely accessible we have included a significant amount of introductory material.

Starting in §III, we review properties of the Heisenberg–Weyl group, its multiplication, nonintegrable phase factor, action upon quantum states and utility as a device for constructing a quantal analogue of classical phase space. In §IV there appears an intuitive overview of our main result. This we develop in §V and §VI. The outcome is a nonlinear quantization prescription involving a  $\psi$ -dependent Hamiltonian operator which then determines the desired wave equation.

The most unfamiliar aspect of this work shall likely be the fact that we seek a quantization process which returns the classical dynamics one started with. The idea seems odd to begin with, but will appear natural in the course of our development. We shall gain via this route a translation of classical mechanics directly into the mathematical framework of quantum mechanics. After §VI, the paper develops properties of the resulting dynamical system, its peculiarities and its surprising connection with the ordinary theory of linear quantum mechanics.

### III. SOME PRELIMINARY MATHEMATICAL OBSERVATIONS

Our result follows from purely geometrical considerations of an elementary nature. In essence we need only the following three sets of observations to obtain the quantization prescription. All of the quoted results are standard. Useful source materials include[17–19] for study of Weyl operators and their relationship to coherent states, and[20, 21] for mathematical aspects of classical and quantal dynamics.

Note that it proves easiest to deduce our results without explicit reference to any particular Hilbert space representation. According to the Stone–von Neumann theorem[22, 23] all irreducible Hilbert space representations of the canonical commutation relations are unitarily equivalent[24]. It then follows that any calculation which one might choose to carry out in a particular Hilbert space representation, but which happens to be predicated purely upon the canonical commutation relations, shall have a precise analogue in any other, for each such representation is merely a special concrete manifestation of a single abstract mathematical object.

#### A. Observations regarding Weyl operators

We shall focus throughout upon a ubiquitous mathematical object known as the Heisenberg–Weyl group[18, 19]. Here we shall deal with quantal systems having only one continuous degree of freedom. The generalization to  $n$  continuous degrees of freedom is straightforward[18].

Let  $\hat{a}^\dagger$  and  $\hat{a}$  denote the usual creation and annihilation operators satisfying the commutation relation  $[\hat{a}, \hat{a}^\dagger] = \hat{I}$ . In terms of these the Weyl operator,  $D(\alpha)$ , is defined to be

$$D(\alpha) \equiv \exp\{\alpha \hat{a}^\dagger - \alpha^* \hat{a}\}, \quad D^\dagger(\alpha) = D(-\alpha), \quad \alpha \in \mathbb{C}.$$

The object looks bleakly abstract to begin with but it is very simple to visualise. It is a unitary operator depending upon a single complex parameter. Although the underlying group has no finite dimensional representation, so that we cannot write it out as a simple matrix, the objects themselves may be viewed one-to-one as residing on a plane. A picture of this variety is invaluable throughout, see Fig. 1.

The commutation relations alone are sufficient to determine the following group multi-

plication rule[18]:

$$D(\alpha)D(\beta) = \exp\{i\text{Im}[\alpha\beta^*]\}D(\alpha + \beta). \quad (1)$$

This simple rule expresses all that we need to know about manipulation of the abstract operators  $D(\alpha)$ . It was discovered by Weyl[16], who gave a simple way to understand it as an encoding of quantal noncommutativity via the presence of the phase factor  $\text{Im}[\alpha\beta^*]$ . What we have in the operators  $D(\alpha)$  is a familiar Abelian group, addition on the complex plane, modified by the inclusion of a leading phase factor. Such a group representation is termed a *projective* or *ray* representation[16]. It would appear that such group representations must enjoy an intimate relationship with the deep subject of quantal phase factors[25, 26].

To understand the importance of the leading phase let us now consider the action of the operators  $D(\alpha)$  upon some Hilbert space that carries a representation of them. For example, consider the vacuum ket  $|0\rangle$  as defined by the annihilation condition  $\hat{a}|0\rangle = 0$ . Of interest shall be the class of translated vacuum states defined by the rule  $|\alpha\rangle \equiv D(\alpha)|0\rangle$ . As is well known[17–19] these happen to be the familiar minimum uncertainty states that were first discovered by Schrödinger. Under further translation, the rule (1) ensures that  $|\beta\rangle$  suffers a simple additive translation of its argument  $\beta$  to  $\alpha + \beta$ :

$$D(\alpha)e^{i\gamma}|\beta\rangle = e^{i(\gamma+\delta\gamma)}|\alpha + \beta\rangle. \quad (2)$$

Here we have included the phase  $\gamma$  to emphasise the nonintegrable nature of the phase developed by an evolving state  $|\alpha\rangle$ . It is related to the quantum geometric phase in a way that we shall later make explicit. To do this we need to understand its origin. It comes directly from the rule (1) as the phase change

$$\delta\gamma = \text{Im}[\alpha\beta^*]$$

Nonintegrability of this phase is clear when we consider a circuit on the group. The quantity

$$D(-\alpha)D(-\beta)D(\alpha)D(\beta) = \exp\{2i\text{Im}[\alpha\beta^*]\}\hat{I}$$

has on the left an expression that describes a circuit on the group manifold, on the right it has a phase factor that multiplies the unit operator. If we were to place a ket to the right of both operators then we get a phase that the ket develops under the circuitous transport dictated by the path on the group. This is not the Berry phase[25], but we can extract that from it. Most important of all the phase change is the same for any chosen ket.

## B. Operator change of variables and the role of $\hbar$

A result of great practical utility concerns the close correspondence between the algebra of creation and annihilation operators and that of the canonically conjugate position and momentum operators. They may be carried one to another by a mere change of variables. A clear recognition of this property is important to understanding the role of  $\hbar$  within this work.

Suppose we were to take  $\hat{p}$  and  $\hat{q}$  as primary. Consider now the new operators

$$\hat{a} = c_q \hat{q} + i c_p \hat{p} \qquad \hat{a}^\dagger = c_q \hat{q} - i c_p \hat{p}.$$

If  $\hat{q}$  and  $\hat{p}$  satisfy the canonical commutation relation  $[\hat{q}, \hat{p}] = i\hbar \hat{I}$  then explicit calculation shows that  $\hat{a}$  and  $\hat{a}^\dagger$  satisfy  $[\hat{a}, \hat{a}^\dagger] = \hat{I}$  provided only that we choose the constants  $c_p$  and  $c_q$  so that  $c_q c_p = 1/2\hbar$ .

In harmonic oscillator problems one uses the standard change of variables

$$c_q = \sqrt{\omega/2\hbar} \qquad c_p = 1/\sqrt{2\hbar\omega},$$

where  $\omega$  has the dimensions of mass by inverse time. If there is no characteristic time then  $\omega$  is free, so it would appear that we cannot exploit the above connection.

Surprisingly, this is not true. In Weyl operator calculations one can generally ignore the constants  $c_q$  and  $c_p$ . To see why one carries out the substitutions

$$\alpha = c_q q + i c_p p \quad \text{and} \quad \hat{\alpha} = c_q \hat{q} + i c_p \hat{p}, \tag{3}$$

to convert  $D(\alpha)$  into its equivalent Weyl form[30]:

$$U[q, p] \equiv \exp\{i/\hbar(p\hat{q} - q\hat{p})\}. \tag{4}$$

It is significant that the  $c_q$  and  $c_p$  dependence now appears only through their product  $c_q c_p = 1/2\hbar$ . It therefore follows that a free translation is possible between both pictures by use of (3) with the ratio  $c_q/c_p$  remaining undetermined. So long as due care is exercised one can set  $c_q = c_p = 1$  and re-insert  $\hbar$  or either constant, via the use of dimensionality considerations alone.



### C. Quantal Phase Space as an Equivalence Class in $\Psi$

Let us now construct a natural quantal analogue of classical phase space, being only a minor variant of that commonly used in the theory of coherent states[30]. Rather than use the Weyl operator labels  $(q, p)$  as the analogue of classical phase space, we prefer to construct phase space as an equivalence class  $\tilde{\psi}(q, p)$  of states  $\psi$  that share the same expectation values for  $\hat{p}$  and  $\hat{q}$ .

First let us review properties of the Weyl group action upon arbitrary kets. Consider the Weyl translates of an arbitrary state labelled  $|\phi\rangle$ . Of interest therefore is the class of states:

$$|q, p; \phi\rangle = U[q, p]|\phi\rangle.$$

As is well known, the linear action of  $U[q, p]$  effects a mere additive translation of the expectation values of any given state  $\phi$ . The reason for this may be traced to the simple relationships[17]:

$$U[q, p]^\dagger \hat{p} U[q, p] = \hat{p} + p\hat{I} \quad \text{and} \quad U[q, p]^\dagger \hat{q} U[q, p] = \hat{q} + q\hat{I}. \quad (5)$$

As a consequence of this fact it is then clear that given any  $\phi$ , one can compute its position and momentum expectation values and then use these parameters in a Weyl operator so as to translate  $\phi$  back to a special representative state  $\phi_0 \equiv U[-\langle\hat{q}\rangle_\phi, -\langle\hat{p}\rangle_\phi]|\phi\rangle$  characterised by the property that both expectation values are equal to zero. In this manner one obtains a convenient parametrisation of the entire Hilbert space  $\mathcal{H}$  in terms of the Weyl translates  $U[q, p]|\phi_0\rangle$  of all states  $\phi_0$  having both expectation values equal to zero. This result is of some utility in picturing the entire Hilbert space as being generated out of all states  $\phi_0$ , see Fig. 2.

Of related utility is the following *coordinate* map

$$\Pi : \mathcal{H} \mapsto \mathbb{R}^2 \quad \text{where} \quad \Pi[\psi] = (q(\psi), p(\psi)); \quad q(\psi) \equiv \langle\hat{q}\rangle_\psi \quad \text{and} \quad p(\psi) \equiv \langle\hat{p}\rangle_\psi. \quad (6)$$

This rule throws away the  $\psi$ -dependence and enables us to extract classical coordinates. It functions here as a mathematical device for defining a quantal analogue of classical phase space as the natural  $\Pi$ -induced equivalence class:

$$\tilde{\psi}(q, p) = \{\psi \in \mathcal{H} : \Pi[\psi] = (q, p) \in \mathbb{R}^2\}. \quad (7)$$

The interpretation of this rule is simple. We partition  $\mathcal{H}$  into all possible sets of states  $\tilde{\psi}(q, p)$  which happen to share the same expectation values for position and momentum operators. A particular class  $\tilde{\psi}(q, p)$  can now be generated via application of  $U[q, p]$  to the special class  $\tilde{\psi}(0, 0)$  consisting of all  $\phi_0$  such that  $\Pi[\phi_0] = (0, 0)$ . This class of states plays the role of an origin in the quantal phase space so constructed[31].

To make the above ideas concrete, we consider now the explicit nature of  $\tilde{\psi}(q, p)$  in the coordinate representation of a one degree of freedom system. To avoid confusion between operators and the group parameters it is helpful to first relabel  $q$  and  $p$  as  $Q$  and  $P$ . Then one can use the definition (4), and an elementary application of the Baker–Campbell–Hausdorff formula[18],  $e^A e^B = e^{1/2[A, B]} e^{A+B}$ , for  $[A, [A, B]] = 0$  and  $[B, [A, B]] = 0$ , so as to disentangle the action of  $\hat{q}$  and  $\hat{p}$ . If at the same time one makes passage to the Schrödinger representation, via the substitutions  $\hat{q} \mapsto q$  and  $\hat{p} \mapsto -i\hbar\partial/\partial q$ , then the relabelled operators  $U[Q, P]$  can be written in either of the two convenient reordered forms:

$$U[Q, P] = \exp\{+iPQ/2\hbar\} \exp\{-Q\partial/\partial q\} \exp\{iPq/\hbar\}, \quad (8)$$

$$U[Q, P] = \exp\{-iPQ/2\hbar\} \exp\{iPq/\hbar\} \exp\{-Q\partial/\partial q\}. \quad (9)$$

Use of the operational calculus identity

$$\exp\{-Q\partial/\partial q\} f(q) = f(q - Q),$$

then enables one to compute, via either route, the general result[18]

$$U[Q, P]\phi(q) = \exp\{-iPQ/2\hbar\} \exp\{iPq/\hbar\} \phi(q - Q) \in \tilde{\psi}(q, p). \quad (10)$$

If one now lets  $\phi_0(q)$  be an arbitrary member of the class  $\tilde{\psi}(0, 0)$ , then it is clear that  $\psi(q) \equiv e^{-iPQ/2\hbar} e^{iPq/\hbar} \phi_0(q - Q)$  is an arbitrary member of the class  $\tilde{\psi}(Q, P)$ .

#### IV. WEYL OPERATORS AS PROPAGATORS

Putting together the preceding material we are led to the following rather striking observation. If one considers an arbitrary classical Hamiltonian system  $H_c(q, p)$  and we pick a particular solution, say  $(q(t), p(t))$ , then it is a pure triviality to notice that the lifting of this trajectory into the argument of a time-dependent Weyl operator  $U[q(t), p(t)]$  creates a propagator which reproduces the classical trajectory as a time-dependent equivalence class

of states  $\tilde{\psi}(q(t), p(t))$ . Moreover, it is a straightforward property of Weyl translation that the resulting evolution must be non-dispersive for all member states of the time-dependent class  $\tilde{\psi}(q(t), p(t))$ .

Therefore, we are led to the non-trivial conclusion that if one can find a way to generate a dynamics of Weyl operators, such that the arguments of these follow the classical trajectories of any chosen classical system, then one will have a wavefunction dynamics for the equivalence classes  $\tilde{\psi}(q, p)$  such that their expectation values also follow the desired classical trajectories. Within such a scheme, one can then see how the macroscopically successful theory of classical mechanics might well just amount to a special form of quantum mechanics where the *equivalence class* is known but not the actual *wavefunction*.

The remarkable adjunct to this conclusion is the knowledge that the Weyl operator action within ordinary linear quantum mechanics is trivial, it is just linear propagation upon the plane. But that very triviality aids us in a totally surprising way. Because of the linear group action, we cannot achieve the desired goal by using a linear theory of quantum mechanics. But the companion of that statement is that because of this very same linear action, there does exist a natural nonlinear theory which does it.

The trick is that any smooth trajectory is built up out of infinitesimal linear segments. As the generator of a given infinitesimal Weyl propagator  $U[\delta q, \delta p]$ , that adds increments  $\delta q$  and  $\delta p$  to the coordinates of some  $\tilde{\psi}(q, p)$ , one might consider the possibility of taking a quantal *phase space dependent* Hamiltonian operator  $\hat{H}(\tilde{\psi})$ . If one were to allow this operator to vary across the quantal phase plane then it is intuitively plausible to suppose that one might build up the right dynamics precisely because of what seemed to be a problematic linear action of the Heisenberg–Weyl group.

## V. A DIRECT GEOMETRIC CONNECTION

To pursue this idea we now need a way to get the required quantal Hamiltonian from knowledge of the classical one. Astonishingly one can do this in a very simple manner by exploiting the geometrical content of classical mechanics.

Recall that the Hamiltonian formulation of classical dynamics defines a vector field for

the flow of phase space points according to the formulæ:

$$\dot{p} = -\frac{\partial H_c}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H_c}{\partial p}, \quad (11)$$

where of course  $H_c(q, p)$  is the classical Hamiltonian. Given a family of trajectories in classical phase space, let us pick one, say  $(q(t), p(t))$ . Having chosen to parametrise this curve with time differentiation of it must yield the local dynamical field  $(\partial H_c/\partial p, -\partial H_c/\partial q)$  as a consequence of (11). A similar property is shared by the evolution operators of quantum mechanics. Given  $U(t, t_0)$  one can deduce  $\hat{H}(t)$  by differentiation and use of the simple rule  $\hat{H}(t) = i\hbar[d/dt U(t, t_0)]U(t, t_0)^{-1}$ . This trick provides the essential geometric connection that we shall need to forge a natural nonlinear quantization condition which provides the required  $\hat{H}(\tilde{\psi})$  in a manner that is unique up to a multiple of the unit operator. The latter freedom shall subsequently be removed via energetic considerations.

## VI. A “CLASSICAL” QUANTIZATION PROCEDURE

Recall that the motion of classical phase space points is to be implemented using elements of the equivalence classes  $\tilde{\psi}(q, p)$ . To transport these in a manner that duplicates the classical motion it is clear that we must lay down upon the plane of parameters of the Weyl operators an operator-valued analogue of the Hamiltonian vector field pertaining to the classical system of interest.

Starting with a family of classical trajectories that are the solutions of a particular classical Hamiltonian  $H_c(q, p)$ , we shall simply project these onto the quantal phase space so as to obtain the required quantal operator-valued field. Throughout, it helps to keep in mind two copies of the phase plane. In the classical plane dynamics is the motion of phase space points; in the quantal plane dynamics is the motion of wavefunctions or operators that happen to be indexed by complex numbers. To underline this we shall initially reserve explicitly complex quantities as referring to the quantal phase plane and explicitly real quantities as referring to the classical phase plane. The situation is depicted in Fig. 3.

The required projection is the standard association

$$c_q q(t) + i c_p p(t) \mapsto \alpha(t),$$

where the parameter  $t$  is time, but  $H_c(q, p)$  need not be explicitly time-dependent and the constants  $c_q$  and  $c_p$  are free.

Working with Weyl operators in the  $D(\alpha)$  form we start by seeking the unique quantal Hamiltonian that generates the following desired group trajectory:

$$\tilde{D}(\alpha(t)) \equiv \exp \left\{ i \int_0^t \text{Im}[\alpha^*(\tau) \dot{\alpha}(\tau)] d\tau \right\} D(\alpha(t)),$$

where  $\alpha(t) = c_q q(t) + i c_p p(t)$  and the leading phase has been chosen purely for convenience, but shall later be fixed via energetic considerations.

In virtue of the linear action of the operators  $D(\alpha)$  one can easily guess that the generator of such a trajectory must be the infinitesimal Weyl operator  $D(\dot{\alpha} \delta t)$ . This guess is supported by using the rule (1) to evaluate the product  $D(\dot{\alpha} \delta t) D(\alpha(t))$  so as to find the expected result

$$D(\dot{\alpha} \delta t) D(\alpha(t)) = \exp\{i \text{Im}[\alpha^* \dot{\alpha}] \delta t\} \times D(\alpha(t) + \dot{\alpha} \delta t). \quad (12)$$

To prove that this guess is correct we need only differentiate the trajectory of the Weyl operator  $\tilde{D}(\alpha(t))$ . From appendix one we obtain the identity

$$\frac{d}{dt} \tilde{D}(\alpha(t)) = (\dot{\alpha} \hat{a}^\dagger - \dot{\alpha}^* \hat{a}) \tilde{D}(\alpha(t)), \quad (13)$$

so that we may now read off the required quantal Hamiltonian. Doing that we discover that  $\hat{H}_q(\alpha)$  is an implicit function of  $\alpha$  given at this stage only in terms of  $\dot{\alpha}$  by the expression:

$$\hat{H}_q(\alpha) = i(\dot{\alpha} \hat{a}^\dagger - \dot{\alpha}^* \hat{a}), \quad (14)$$

To remove the implicit nature of this connection, we now use the correspondence constraint,  $\alpha = c_q q + i c_p p$ , its derivative  $\dot{\alpha} = c_q \dot{q} + i c_p \dot{p}$ , and the fact that the validity of Hamilton's equations for the original classical system ensures that

$$(\dot{q}, \dot{p}) = (\partial H_c / \partial p, -\partial H_c / \partial q).$$

Solving for  $\dot{\alpha}$  one discovers the quantization condition:

$$\hat{H}_q(\alpha) = i \left( c_q \frac{\partial H_c}{\partial p} - i c_p \frac{\partial H_c}{\partial q} \right) \hat{a}^\dagger + \left( c_q \frac{\partial H_c}{\partial p} + i c_p \frac{\partial H_c}{\partial q} \right) \hat{a}, \quad (15)$$

where it should be stressed that the the derivatives of the classical Hamiltonian are to be evaluated upon the quantal phase plane at the point specified by the rule  $(q, p) \leftrightarrow \alpha = c_q q + i c_p p$

The relation (15) thus fixes  $\hat{H}_q(\alpha)$  in terms of the projected derivatives of the classical Hamiltonian from the classical phase plane to the quantal phase plane, at the corresponding

point and at the corresponding time. Our quantization condition is thus uniquely derived, modulo questions of phase arbitrariness, by the purely geometric constraint that the Weyl operator dynamics should reproduce the classical dynamics for *all possible*  $H_c(q, p)$ .

It is helpful to now drop the convention that quantal phase plane quantities should be complex. The algebraic changes:  $\alpha = c_q q + i c_p p$  and  $\hat{a} = c_q \hat{q} + i c_p \hat{p}$  with  $c_q c_p = 1/2\hbar$ , now convert  $\tilde{D}(\alpha(t))$  into its equivalent form

$$\tilde{U}[q(t), p(t)] = \exp \left\{ \frac{-i}{2\hbar} \int_0^t (p(\tau) \dot{q}(\tau) - q(\tau) \dot{p}(\tau)) d\tau \right\} U[q(t), p(t)], \quad (16)$$

while equations (13) and (15) become:

$$\frac{d}{dt} \tilde{U}[q(t), p(t)] = \frac{-i}{\hbar} (\dot{q} \hat{p} - \hat{p} \dot{q}) \tilde{U}[q(t), p(t)], \quad (17)$$

$$\hat{H}_q(q, p) = \frac{\partial H_c}{\partial q}(q, p) \hat{q} + \frac{\partial H_c}{\partial p}(q, p) \hat{p}. \quad (18)$$

In this manner the choice of scale in both the original classical system and that of its formal quantal correspondent is made identical. Moreover, it is readily seen that a change in  $\hbar$  has no effect other than that of rescaling all classical trajectories.

## VII. DISCUSSION OF THE RESULT

The existence of this remarkable dynamical correspondence depends upon the highly nontrivial fact that the Weyl group action upon the phase plane is linear. Because of this fact the “tangent” vectors to the trajectories in both classical mechanics and this special Weyl operator dynamics are just the derivatives of the phase plane parameters. Since the two parameter spaces are isomorphic, and the depth of Hamilton’s equations is just the statement that  $H_c(q, p)$  specifies how the field of tangent vectors should vary across the phase plane, we are able to achieve a direct quantization that generates an isomorphic dynamics of Weyl operators.

For those concerned about our use of the time  $t$ , let it be noted that the geometric nature of our argument ensures that  $t$  appears in this derivation purely as a parameter that enforces a synchronicity between two identical families of trajectories. It ties the temporal development of individual points in the classical and quantal planes one-to-another. That is why we are able to both use it and then later remove it. Therefore, it should be stressed that there are no assumptions of adiabaticity, or anything of that nature. If the initial classical

Hamiltonian were to contain an explicit time dependence then the right hand side of (15) would also be explicitly time-dependent and would so fix an operator valued field that varies over the group in time.

A direct view of the result is offered upon consideration of the small time propagator argument that motivated it. Using (18) in the small time Weyl propagator:

$$U_{\delta t}[q, p] \equiv \exp \left\{ \frac{-i}{\hbar} \left[ \frac{\partial H_c}{\partial q} \hat{q} + \frac{\partial H_c}{\partial p} \hat{p} \right] \delta t \right\},$$

one computes the new group element

$$U_{\delta t}[q, p] \cdot \tilde{U}[q, p] = \exp \left\{ \frac{-i}{2\hbar} \left[ p \frac{\partial H_c}{\partial p} + q \frac{\partial H_c}{\partial q} \right] \delta t \right\} U[q + \frac{\partial H_c}{\partial p} \delta t, p - \frac{\partial H_c}{\partial q} \delta t]$$

which must be  $\tilde{U}[q(t + \delta t), p(t + \delta t)]$ . So having carried the classical dynamical field down to the quantal phase plane, one sees directly that the integration of the equation of motion for our propagator corresponds precisely to an integration of the original classical equations in Hamiltonian form.

Note that the well known quantum-classical dynamical correspondence[19] for Hamiltonians which are quadratic or less in  $\hat{q}$  and  $\hat{p}$  is separate from our result. However, it is worthy of note that an early paper by Klauder[27], upon the action formulation of quantum dynamics, passes very close by the discovery made here that a complete quantum to classical dynamical correspondence is possible once we permit a nonlinear Schrödinger equation involving a state-dependent Hamiltonian. There it was shown that one can reproduce the dynamics of the Harmonic oscillator using only a linear combination of the operators  $\hat{q}$  and  $\hat{p}$ .

On a deeper level, our entire result depends upon the remarkable mathematical fortuity that the Heisenberg-Weyl group affords a ray representation of the Abelian group of translations on the complex plane[16]. As such we have uncovered something which is mathematically obvious, but perhaps physically unexpected.

## VIII. A “CLASSICAL” SCHRÖDINGER EQUATION

Thus far we have been concerned with dynamics upon the Heisenberg–Weyl group. This must now be carried to the quantum states.

Combining equations (17) and (18) we obtain the general Heisenberg–Weyl group evolution equation

$$i\hbar \frac{d}{dt} \tilde{U}[q, p] = \hat{H}_q(q, p) \tilde{U}[q, p]; \quad \text{where} \quad \hat{H}_q(q, p) = \frac{\partial H_c}{\partial q}(q, p) \hat{q} + \frac{\partial H_c}{\partial p}(q, p) \hat{p} \quad (19)$$

where the coordinates have meaning with respect to the group manifold. Given this abstract dynamics, it is clear that we can place an arbitrary quantum state  $\psi$  to the right, and so obtain an evolution equation for quantum states. The problem is then how to fix  $(q, p)$  from knowledge of  $\psi$ .

Since  $\tilde{U}[q(t), p(t)]$  simply translates expectation values, it is clear that one must carry the abstract Weyl dynamics to the quantum states  $\psi$  by fixing  $(q, p)$  in the operator  $\hat{H}_q(q, p)$  in terms of the coordinates

$$q(\psi) = \langle \hat{q} \rangle_\psi \quad \text{and} \quad p(\psi) = \langle \hat{p} \rangle_\psi \quad (20)$$

of the equivalence class  $\tilde{\psi}(q, p)$  to which  $\psi$  belongs.

Doing that we arrive at the formal nonlinear Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H}_q(\tilde{\psi}) |\psi\rangle. \quad (21)$$

It is important to understand that this prescription is arrived at in order that our dynamical correspondence construction should work. There is no empirical evidence for its physical validity but the possibility appears worthy of some attention.

### A. A generalized dynamical law

In this respect, it is remarkable that a dynamics defined by (21) is always norm-preserving provided only that  $\hat{H}_q(\tilde{\psi}) = \hat{H}_q^\dagger(\tilde{\psi}^*)$ . To prove this one uses (21) and its conjugate to find that

$$i\hbar \frac{d}{dt} \langle \psi | \psi \rangle = \langle \dot{\psi} | \psi \rangle + \langle \psi | \dot{\psi} \rangle = \langle \psi | \left[ -\hat{H}_q^\dagger(\tilde{\psi}^*) + \hat{H}_q(\tilde{\psi}) \right] | \psi \rangle = 0.$$

Thus  $||\psi||^2$  remains an invariant. However, unlike linear quantum theory the Hilbert space inner product  $\langle \phi | \psi \rangle$  is not. The reason for this lies in a repeat of the above argument, where



one is led to discover that

$$i\hbar \frac{d}{dt} \langle \phi | \psi \rangle = \langle \phi | \left[ -\hat{H}_q^\dagger(\tilde{\phi}^*) + \hat{H}_q(\tilde{\psi}) \right] | \psi \rangle \neq 0.$$

In understanding this fact it is instructive to note that the presence of the  $\tilde{\psi}$ -dependence is formally equivalent to the case of a time-dependent Hamiltonian, albeit one such that different states  $\phi$  and  $\psi$  evolve under different Hamiltonians.

The loss of an invariant inner product means that we have now lost much of the Hilbert space structure of linear quantum mechanics. The dynamical arena is now the Banach space[29] of all square integrable  $\psi$ , since we retain an invariant norm  $||\psi||^2$ . This loss of structure does not however mean that the usual Hilbert space methods are useless. One can still use the device of referring the evolution of a given state  $\psi$  with respect to a static orthonormal basis in an auxilliary space[28].

Given a countable orthonormal basis  $\{\phi_k\}_{k=1}^\infty$  one can write any  $\psi$  in terms of the coordinate functionals  $C_k[\psi(t)] \equiv \langle \phi_k | \psi(t) \rangle$ . The intuitive meaning of a nonlinear operator on such a space is then that of an infinite number of nonlinear relations  $C'_k(C_1, C_2, \dots)$  among the initial coordinates  $C_k$  and the final coordinates  $C'_k$ . Knowledge of the  $C'_k$  then enables reconstruction of the transformed state  $\psi'$ . However, it is clear that such an idea will not work in general because convergence of the new norm  $C'_k C''^*_k$  is not assured. Here we are fortunate in that the bootstrap from linear Hermitian generators in the construction of (21) assures us that this will be so. This simple fact means that one might legitimately expect (21) to have a rich mathematical theory. Having said that, difficult problems remain in the elucidation of conditions under which the  $\psi$ -dependence must enter into  $\hat{H}_q(\tilde{\psi})$ .

## B. Minimal invariance

As a temporary measure, it seems appropriate to introduce a concept we call *minimal invariance*. This means invariance of the equations of motion with respect to the ordinary globally linear unitary transformations of quantum mechanics. The idea is that (21) should, at the very least, share the representation independence of linear quantum theory.

Consider therefore the action of a linear operator  $U$  in the equation (21). The suggested transformation law is:

$$|\psi\rangle \mapsto U|\psi\rangle \quad \text{and} \quad \hat{H}_q(\tilde{\psi}) \mapsto U\hat{H}_q(U\tilde{\psi})U^\dagger, \quad (22)$$

where the essential new feature occurs in the carrying of  $U$  to the argument of  $\hat{H}_q(\tilde{\psi})$ .

Examination of this condition, and the Hermiticity constraint  $\hat{H}_q(\tilde{\psi}) = \hat{H}_q^\dagger(\tilde{\psi}^*)$ , is then suggestive of the need to have  $\psi$  enter into  $\hat{H}_q(\tilde{\psi})$  purely via quantities that involve  $\psi$  and  $\psi^*$  in a real-valued and representation independent combination. Expectation values, such as the coordinates (20) of our model nonlinear system, have this property so that minimal invariance is assured in this case. For example, application of (22) to the quantization rule (18) leaves the arguments of the coefficients  $\partial H_c/\partial q$  and  $\partial H_c/\partial p$  invariant, and hence their values, while taking  $\hat{q}$  and  $\hat{p}$  into the appropriately transformed operators  $U\hat{q}U^\dagger$  and  $U\hat{p}U^\dagger$ .

As a finite dimensional example of the same idea, one might consider an ordinary linear Hamiltonian, with the spectral resolution:

$$\hat{H} = \sum_{k=1}^n E_k |\phi_k\rangle\langle\phi_k|.$$

This can be made nonlinear in the sense of (21) by the device of multiplying the eigenvalues by representation independent quantities. An interesting example of such a system is the model:

$$\hat{H}(\psi) \equiv \sum_{k=1}^n E_k (|\langle\phi_k|\psi\rangle|^2) \cdot |\phi_k\rangle\langle\phi_k|.$$

One has on the space of states  $\psi \in \mathbb{CP}^{n-1}$  a smoothly varying operator field  $\hat{H}(\psi)$ , such that the original stationary states  $\phi_k$  remain, as do the energy eigenvalues  $E_k$ . However, away from these unperturbed stationary states the dynamics is quite different. It remains completely integrable, but now includes shear motion.

### C. A possible physical interpretation

One formal way to view the dynamical law (21) is as the natural generalization of linear quantum mechanics under the postulate that *global unitarity* in function space should be relaxed into the wider system of *local unitarity*, where the intended concept of locality refers to points in the abstract topological space of all possible  $\psi$ . There is no physical motivation for that view other than as an exploratory route for the purpose of generalization.

A more physical approach would be to interpret a state-dependent Hamiltonian as standing for elements of back-reaction of a system upon its environment. If  $\psi$  is all, if  $\psi$  is a real entity; then it ought to be the case that the value of  $\psi$  for a large and non-isolated system

should determine in some appropriate sense its future evolution and hence its Hamiltonian. An approach of this kind is reminiscent of Everett's Universal  $\Psi$ [34]. On another tack, it is considered most encouraging that it is precisely within the classical regime, that a dynamics of the type (21) finds correspondence with a successful physical theory.

However, it appears premature to devote too much attention to the general rule (21). Many questions of principle remain to be solved. The rest of this paper is therefore concerned with refining understanding of the particular system defined by (18) rather than general consideration of (21). As a final remark on that topic it is interesting that the general theory of Weinberg[8, 9] shares the property of norm-preserving dynamics. It seems that a completely general theory based upon (21) would have to be equivalent to that of Weinberg[32].

## IX. CORRECTING THE ENERGY

Since our geometrical construction retains expectation values for the coordinates, we make the postulate that these remain the correct way to extract the values of general state-dependent observables. To the operator  $\hat{A}(\tilde{\psi})$  we then associate the observable value:

$$A(\tilde{\psi}) = \langle \psi | \hat{A}(\tilde{\psi}) | \psi \rangle. \quad (23)$$

It is remarkable that this postulate works, most especially so in respect of the well-known observation that the standard probabilistic interpretation of  $\psi$  cannot hold in the absence of an invariant Hilbert space inner product[33]. Presently, it is not clear to us that this fact should rightly be considered a *problem* or perhaps a *potential solution to a problem*; particularly so in connection with the deep unresolved problems of quantum measurement theory[3, 4].

Our postulated rule (23) therefore has merit in so far as it is successful for the purpose of constructing a quantal model of classical mechanics. Moreover, it appears necessary in order to achieve consistency with our use of the coordinate rule (20).

Using (23) and the operator (18) we can now compute the energy of our model classical system when in the state  $\psi$ :

$$E_0(\psi) = \frac{\partial H_c}{\partial q}(q, p) \cdot \langle \hat{q} \rangle_\psi + \frac{\partial H_c}{\partial p}(q, p) \cdot \langle \hat{p} \rangle_\psi; \quad (q, p) = (\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi). \quad (24)$$

This is certainly not the desired exact classical energy.

However, a most interesting possibility now presents itself. If one recalls that there was an essential time-dependent phase arbitrariness in the mapping to Weyl operator trajectories, then there appears an avenue towards the dual resolution of two problems. Let us therefore explore the idea that a removal of the quantal phase freedom might be connected with the necessity of correcting the classical energy!

To pursue an energetic correspondence with classical mechanics we now invoke the remaining freedom to add a multiple of the unit operator to the Hamiltonian  $H(\tilde{\psi})$  as the sole available means of altering the value of our energy observable,  $E(\tilde{\psi})$ , without affecting the desired classical to quantal dynamical correspondence.

Since both the desired energy,  $E(\tilde{\psi}) = H_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi)$ , and the original energy (24) are pure numbers, we need only take their difference so as to construct the correction operator:

$$\left[ H_c - \frac{\partial H_c}{\partial q} \cdot \langle \hat{q} \rangle_\psi - \frac{\partial H_c}{\partial p} \cdot \langle \hat{p} \rangle_\psi \right] \hat{I}, \quad (25)$$

as being that unique multiple of the unit operator which we must add to the original quantization condition (18).

The residual phase arbitrariness of the original derivation is thereby removed and we arrive at a uniquely determined quantization condition:

$$\hat{H}_q(\tilde{\psi}) \equiv H_c(\tilde{\psi}) + \frac{\partial H_c}{\partial p}(\tilde{\psi})(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial H_c}{\partial q}(\tilde{\psi})(\hat{q} - \langle \hat{q} \rangle_\psi). \quad (26)$$

In pausing to consider this situation, we find it quite remarkable that the entire procedure is successful to this degree of correspondence. Not only have we recovered exact classical dynamics, but also the correct energy. Moreover all of this is achieved with a seemingly perfect blend between both classical and quantal concepts. Although some scepticism about the postulates made appears warranted, we find it truly incredible that a construction having this level of completeness is even possible.

## X. EHRENFEST'S THEOREM

As a prelude to discussion of Poisson brackets, commutators and the inclusion of classical canonical transformations, it is instructive to verify that the prescription (26) actually does recover classical dynamics in Hamiltonian form. The discussion exposes a rather special property of our quantal phase space of equivalence classes.

We begin by making an assumption about the quantal phase space coordinates  $\langle \hat{q} \rangle_\psi$  and  $\langle \hat{p} \rangle_\psi$ . Let us suppose initially that the operators  $\hat{q}$  and  $\hat{p}$  do not depend upon  $\psi$ . As a result of this assumption it follows that:

$$\frac{d}{dt} \langle \hat{q} \rangle_\psi = \langle \dot{\psi} | \hat{q} | \psi \rangle + \langle \psi | \hat{q} | \dot{\psi} \rangle, \quad (27)$$

$$\frac{d}{dt} \langle \hat{p} \rangle_\psi = \langle \dot{\psi} | \hat{p} | \psi \rangle + \langle \psi | \hat{p} | \dot{\psi} \rangle. \quad (28)$$

Using this decomposition, we now invoke the defining equation of motion (21) and the quantization rule (26) to discover that

$$\frac{d}{dt} \langle \hat{q} \rangle_\psi = \frac{1}{i\hbar} \left\langle [\hat{q}, \hat{H}_q(\tilde{\psi})] \right\rangle = \frac{\partial H_c}{\partial p}(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi), \quad (29)$$

$$\frac{d}{dt} \langle \hat{p} \rangle_\psi = \frac{1}{i\hbar} \left\langle [\hat{p}, \hat{H}_q(\tilde{\psi})] \right\rangle = -\frac{\partial H_c}{\partial q}(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi). \quad (30)$$

Thus we recover Hamilton's equations, and a special version of Ehrenfest's theorem[36] such that the expectation value angle brackets now lie within the argument of the Hamiltonian derivatives, rather than surrounding them.

### A. The standard classical limit

Standard proofs of the classical limit[36] involve an argument designed to recover the above couplet of equations in the limit  $\hbar \rightarrow 0$ , while making assumptions about the states  $\psi$  and the behaviour of  $\partial H_c / \partial q$ . With that focus it seems that people either neglected to notice, or did not deem it important, that the above desired limit holds irrespective of the value of  $\hbar$ . It thus permits an exact reproduction of classical mechanics for a fixed non-zero value of  $\hbar$ . The importance this work is that we have now provided a natural route for deduction of the appropriate quantization prescription (26), where the chosen route *does not require* us to know the ordinary quantization prescription of linear quantum theory.

In this connection, it should be noted that our examination of the literature has uncovered a recent reformulation of linear quantum theory due to Kay[37], wherein the appropriate quantization condition (26) appears in the course of an argument he uses to establish the standard classical limit. A similar result is implicit in Messiah's classic book[47]. However, in both cases one starts from knowledge of the ordinary quantization prescription and the connection with nonlinear quantum theory is not recognised.

Later we shall show that continuation of the rule (26) to all orders in the implicit Taylor series enables recovery of the usual quantization prescription of linear quantum theory. It then follows that the classical limit may now be viewed in a completely  $\hbar$ -independent fashion as the truncation of an operator Taylor series at the first-order term. In that view appropriate to the current orthodoxy, classical mechanics thus appears as a dynamically *nonlinear* approximation to *linear* quantum theory.

## B. The case of nonlinear coordinate observables

Our preceding derivation was predicated upon the assumption of coordinate observables derived from linear operators. As a primer for discussion of Poisson brackets it is therefore useful to consider the more general case of coordinate observables defined by the implicit relations:

$$\langle Q_c \rangle = Q_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi) \quad \text{and} \quad \langle P_c \rangle = P_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi).$$

This choice is motivated by consideration of the freedom we have to make an arbitrary canonical transformation in the original classical system after its quantization via the rule (26). Since we deal with a finite canonical transformation,  $Q_c(q, p)$  and  $P_c(q, p)$  are not arbitrary. To preserve the Poisson structure they must satisfy the essential condition

$$\{Q_c, P_c\} = \frac{\partial Q_c}{\partial q} \frac{\partial P_c}{\partial p} - \frac{\partial Q_c}{\partial p} \frac{\partial P_c}{\partial q} = 1, \quad (31)$$

where  $\{\bullet, \bullet\}$  denotes the usual Poisson bracket of classical mechanics.

Considering the evolution of  $\langle Q_c \rangle$  and  $\langle P_c \rangle$  one now finds that:

$$\frac{d}{dt} \langle Q_c \rangle = \frac{\partial Q_c}{\partial q} \cdot \frac{d}{dt} \langle \hat{q} \rangle_\psi + \frac{\partial Q_c}{\partial p} \cdot \frac{d}{dt} \langle \hat{p} \rangle_\psi \quad (32)$$

$$\frac{d}{dt} \langle P_c \rangle = \frac{\partial P_c}{\partial q} \cdot \frac{d}{dt} \langle \hat{q} \rangle_\psi + \frac{\partial P_c}{\partial p} \cdot \frac{d}{dt} \langle \hat{p} \rangle_\psi \quad (33)$$

It is now possible to combine this with equations (29) and (30) so as to compute the new relationships:

$$\frac{d}{dt} \langle Q_c \rangle = \langle \{Q_c, H_c\} \rangle \quad (34)$$

$$\frac{d}{dt} \langle P_c \rangle = \langle \{P_c, H_c\} \rangle \quad (35)$$

where the appearance of the classical Poisson bracket is most significant.

In general, the operators  $\hat{Q}$  and  $\hat{P}$  associated with  $\langle Q_c \rangle$  and  $\langle P_c \rangle$  must now depend upon  $\tilde{\psi}$ . However, working backwards, it becomes clear they can be given explicitly in terms of the linear operators  $\hat{q}$  and  $\hat{p}$ , via the rule (26), as the two expressions:

$$\hat{Q}_q(\tilde{\psi}) \equiv Q_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi) + \frac{\partial Q_c}{\partial p}(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial Q_c}{\partial q}(\hat{q} - \langle \hat{q} \rangle_\psi), \quad (36)$$

$$\hat{P}_q(\tilde{\psi}) \equiv P_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi) + \frac{\partial P_c}{\partial p}(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial P_c}{\partial q}(\hat{q} - \langle \hat{q} \rangle_\psi). \quad (37)$$

Considering these nonlinear operators, it is instructive to compute their commutator in the ordinary sense of linear quantum theory:

$$\begin{aligned} [\hat{Q}_q(\tilde{\psi}), \hat{P}_q(\tilde{\psi})] &= \frac{\partial Q_c}{\partial q} \frac{\partial P_c}{\partial p} [\hat{q}, \hat{p}] + \frac{\partial Q_c}{\partial p} \frac{\partial P_c}{\partial q} [\hat{p}, \hat{q}] \\ &= \left( \frac{\partial Q_c}{\partial q} \frac{\partial P_c}{\partial p} - \frac{\partial Q_c}{\partial p} \frac{\partial P_c}{\partial q} \right) [\hat{q}, \hat{p}] \\ &= i\hbar \{Q_c, P_c\}(\tilde{\psi}) \hat{I} \end{aligned}$$

Preservation of the canonical commutation relations at each point in quantal phase space is thus equivalent to the familiar condition (31) that the functions  $Q_c$  and  $P_c$  should generate a finite classical canonical transformation.

Arguing in purely physical terms, one now sees that a decision to work from linear operators  $\hat{q}$  and  $\hat{p}$  involves no loss of generality. The case of nonlinear operators merely signifies our freedom to carry out any desired classical canonical transformation prior to quantization via the rule (26). On a much deeper level this freedom may be traced to the special property of quantal phase space that all equivalence classes  $\tilde{\psi}(q, p)$  can be generated out of any single representative class. The nonlinear operators defined above are therefore merely ordinary linear operators acting upon an automorphism of the original label space  $(q, p) \in \mathbb{R}^2$ . One might just as well choose to carry out that automorphism upon the abstract label space of states as a prelude to carrying the Weyl group action to this space. At the deepest level function space has no preferred coordinatization.

## XI. POISSON BRACKETS AND CANONICAL TRANSFORMATIONS

The previous argument is suggestive of a general role for Poisson brackets. To establish that this is the case let us now work directly from the rules (21) and (26). To any pair of

classical phase space functions  $f_c(q, p)$  and  $h_c(q, p)$  we assign the operators:

$$\hat{f}_q(\tilde{\psi}) \equiv f_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi) + \frac{\partial f_c}{\partial p}(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial f_c}{\partial q}(\hat{q} - \langle \hat{q} \rangle_\psi) \quad (38)$$

$$\hat{h}_q(\tilde{\psi}) \equiv h_c(\langle \hat{q} \rangle_\psi, \langle \hat{p} \rangle_\psi) + \frac{\partial h_c}{\partial p}(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial h_c}{\partial q}(\hat{q} - \langle \hat{q} \rangle_\psi) \quad (39)$$

whose observable values, in the sense of the rule (23), clearly agree with the values of the original classical functions for all  $\psi$ .

We wish now to examine the general role of the usual quantal commutator. To do that one must work from the postulate (21). Before doing so it is helpful to notice that although

$$[\hat{f}_q(\tilde{\psi}), \hat{h}_q(\tilde{\psi})] = i\hbar\{f_c, h_c\}(\tilde{\psi}) \cdot \hat{I},$$

it happens that

$$i\hbar\{f_c, h_c\}(\tilde{\psi}) \cdot \hat{I} \neq i\hbar\{f_c, h_c\}_q(\tilde{\psi}).$$

Thus the quantization, according to (26), of the classical Poisson bracket is not equal to the commutator of the quantizations of the original classical functions. This is a very important observation in respect of the Groenwold–van Hove theorem[50] which asserts the non–existence of a quantization prescription having the property that it should preserve the Poisson bracket.

However, it is true in general that:

$$\langle [\hat{f}_q(\tilde{\psi}), \hat{h}_q(\tilde{\psi})] \rangle = i\hbar\langle \{f_c, h_c\}_q(\tilde{\psi}) \rangle.$$

A consequence of this peculiarity is that there does not appear to be any simple analogue of the Heisenberg equation of motion for *operators*. However, there is a Heisenberg equation of motion for expectation values.

To obtain this, suppose that

$$i\hbar \frac{d}{dt}|\psi\rangle = \hat{h}_q(\tilde{\psi})|\psi\rangle \quad (40)$$

determines  $\psi(t)$  and consider the expression

$$\frac{d}{dt}\langle \psi | \hat{f}_q(\tilde{\psi}) | \psi \rangle = \langle \dot{\psi} | \hat{f}_q(\tilde{\psi}) | \psi \rangle + \langle \psi | \dot{\hat{f}}_q(\tilde{\psi}) | \psi \rangle + \langle \psi | \frac{d\hat{f}_q(\tilde{\psi})}{dt} | \psi \rangle.$$

Application of (40) now reduces this to

$$\frac{d}{dt}\langle \psi | \hat{f}_q(\tilde{\psi}) | \psi \rangle = \frac{1}{i\hbar}\langle [\hat{f}_q(\tilde{\psi}), \hat{h}_q(\tilde{\psi})] \rangle + \langle \psi | \frac{d\hat{f}_q(\tilde{\psi})}{dt} | \psi \rangle,$$



as a general result valid for any  $\tilde{\psi}$ -dependence. The problem lies now within the final term. To evaluate this we need to know how  $\tilde{\psi}$  enters. Using the definition (39) one can readily compute the result

$$\begin{aligned} \frac{d\hat{f}_q(\tilde{\psi})}{dt} &= \left( \frac{\partial f_c}{\partial p} \frac{d\langle\hat{p}\rangle_\psi}{dt} + \frac{\partial f_c}{\partial q} \frac{d\langle\hat{q}\rangle_\psi}{dt} \right) - \left( \frac{\partial f_c}{\partial p} \frac{d\langle\hat{p}\rangle_\psi}{dt} + \frac{\partial f_c}{\partial q} \frac{d\langle\hat{q}\rangle_\psi}{dt} \right) \\ &\quad + \left( \frac{\partial^2 f_c}{\partial^2 p} (\hat{p} - \langle\hat{p}\rangle_\psi) + \frac{\partial^2 f_c}{\partial p \partial q} (\hat{q} - \langle\hat{q}\rangle_\psi) \right) \frac{d\langle\hat{p}\rangle_\psi}{dt} \\ &\quad + \left( \frac{\partial^2 f_c}{\partial q \partial p} (\hat{p} - \langle\hat{p}\rangle_\psi) + \frac{\partial^2 f_c}{\partial^2 q} (\hat{q} - \langle\hat{q}\rangle_\psi) \right) \frac{d\langle\hat{q}\rangle_\psi}{dt}. \end{aligned}$$

Computing the expectation value of this result, one finds that it is identically equal to zero, whatever the evolution of  $\psi$ .

Thus we obtain the rather striking result

$$\frac{d}{dt} \langle \hat{f}_q(\tilde{\psi}) \rangle = \langle \frac{1}{i\hbar} [\hat{f}_q(\tilde{\psi}), \hat{h}_q(\tilde{\psi})] \rangle = \langle \{f_c, h_c\}(\tilde{\psi}) \rangle, \quad (41)$$

where it should be noted that expectation values are essential. The result (41) fails as an equality for operators.

In this manner one sees that the nonlinear quantization condition derived at (26) actually recovers Dirac's formal rule of replacement[45] in the modified form

$$\langle \{\bullet, \bullet\} \rangle \rightarrow \langle [\bullet, \bullet] / i\hbar \rangle.$$

The inclusion of the angle brackets now makes this an exact result which depends in no way upon the magnitude of  $\hbar$ , but which applies only in the case of a quantization procedure of the form (26).

It follows from (41) that the dynamical system defined by (21) and (26) enjoys a complete inclusion of the full group of classical symmetries. Any  $\hat{h}_q(\tilde{\psi})$  can be now considered as the generator of a one parameter group of symplectic diffeomorphisms on quantal phase space[20], the canonical transformations of Hamiltonian classical mechanics. However, there now occurs, at least within this special dynamical system, the additional freedom to include the full group of nonlinear canonical transformations rather than just the linear ones as are currently employed in the traditional phase space formulation of quantum mechanics[48].

## XII. MEASUREMENT THEORY

As one might anticipate, there are some difficult conceptual problems associated with the introduction of ordinary measurement theoretic concepts to this unusual version of classical mechanics.

### A. Dispersion and the Uncertainty Principle

For instance, Heisenberg's uncertainty principle

$$\langle(\Delta\hat{q})^2\rangle\langle(\Delta\hat{p})^2\rangle \geq \hbar/4, \quad (42)$$

where  $\langle(\Delta\hat{q})^2\rangle \equiv \langle(\hat{q} - \langle\hat{q}\rangle)^2\rangle$  and  $\langle(\Delta\hat{p})^2\rangle \equiv \langle(\hat{p} - \langle\hat{p}\rangle)^2\rangle$ , must remain a valid mathematical statement about wavefunctions and operators.

However, the role of (42) is very unclear within this framework. The difficulty is that the quantization rule (26) does not permit one to go beyond operators that are first-order in  $\hat{p}$  and  $\hat{q}$ . It seems that such problems are mainly conceptual, but it is deemed important to highlight them. It seems that the problems posed by the existence of an embedding of classical mechanics within a generalized quantum theory must provide an excellent focus for deeper investigation of the orthodox epistemological framework.

Given the above observation, one might now pursue the idea of taking the following definition of a generalized operator dispersion:

$$\langle(\Delta\hat{q})^2(\tilde{\psi})\rangle \equiv \langle\hat{q}^2(\tilde{\psi})\rangle - \langle\hat{q}(\tilde{\psi})\rangle^2 \quad \text{and} \quad \langle(\Delta\hat{p})^2(\tilde{\psi})\rangle \equiv \langle\hat{p}^2(\tilde{\psi})\rangle - \langle\hat{p}(\tilde{\psi})\rangle^2. \quad (43)$$

It is not clear that this is the only definition possible, but it seems natural enough. In making the above choice we are merely exploring the question of what can go wrong if one tries to carry over other features of quantum measurement theory, rather than just the rule (23).

Having made the above choice, there now occurs a rather surprising phenomenon. Noting that  $q^2$  and  $p^2$  are quantized under the rule (26) as the nonlinear operators

$$\hat{q}^2(\psi) = \langle q \rangle_\psi^2 + 2\langle q \rangle_\psi(\hat{q} - \langle q \rangle_\psi) \quad \text{and} \quad \hat{p}^2(\psi) = \langle p \rangle_\psi^2 + 2\langle p \rangle_\psi(\hat{p} - \langle p \rangle_\psi),$$

one discovers immediately that

$$\langle\psi|\hat{q}^2(\psi)|\psi\rangle = (\langle\psi|q|\psi\rangle)^2 \quad \text{and} \quad \langle\psi|\hat{p}^2(\psi)|\psi\rangle = (\langle\psi|p|\psi\rangle)^2,$$

from which it follows that

$$\langle (\Delta \hat{q})^2(\tilde{\psi}) \rangle = 0 \quad \text{and} \quad \langle (\Delta \hat{p})^2(\tilde{\psi}) \rangle = 0.$$

So the choice of the rule (43) leads to the curious situation that we have a theory with finite  $\hbar$  in a regime of quantization that appears to have no operator dispersion!

## B. Nonlinearity and the projection postulate

This fact clearly signals that there will be severe problems in attempting to carry over standard measurement theory to our nonlinear theory. The source of the problem appears to lie with the projection postulate of ordinary linear quantum theory. This is implicit in the interpretation of dispersion.

According to this rule, the probability of a state  $\psi$  making a stochastic transition to the state  $\phi$ , the eigenstate of an instantaneous Hamiltonian, is given by:

$$p(\phi|\psi) = |\langle \phi|\psi \rangle|^2. \tag{44}$$

Invocation of this rule requires that  $|\langle \phi|\psi \rangle|^2$  be an invariant. Currently it is generally argued[33] that this alone rules out the possibility of nonlinear evolution. Central to that argument is Wigner's theorem, which shows that the only continuous probability preserving automorphisms of the space of states are the linear unitary transformations. Hence it is commonly asserted that quantal evolution must *always* be linear.

The primary difficulty with this conceptual standpoint is that the stochastic process  $\psi \mapsto \phi$  according to the rule (44) is manifestly not a linear unitary process. If there is no transition or measurement, the possibility of one, and the fact that (44) gives the correct probabilities, certainly demands, via Wigner's theorem, that the evolution be linear in that situation[33]. However, this argument does not demand that quantal evolution be linear *at all times*. In order to invoke Wigner's theorem, we must assume that the quantity (44) has a physical meaning. In doing so, we accept that there is a phase of quantal evolution that is not linear and which has the stochastic description given by (44). It is the possibility of this *special* mode of evolution which enforces linearity whenever the quantum state  $\psi$  is not undergoing a process described by (44) and where we wish to allow that it might. This subtlety of the argument is rarely appreciated. When it is[33], it is usually flagged by

stating that linearity is enforced whenever the system in question is *isolated*. Often that proviso is ignored.

Ordinarily, one now lives with the dichotomy by saying that quantum mechanics is mighty peculiar and one must therefore accept that there be two processes; one which is deterministic, and linear, the other which is stochastic and manifestly nonlinear[3]. However, if we desire a consistency of dynamics then there appears to be a clean split and the two rules do not rest well with one another[39].

When we consider nonlinear quantum theory, the interesting thing is that (44) can no longer be valid in the sense that it is used now. This situation arises because the Hilbert space inner product is no longer a dynamical invariant. We cannot then reserve the possibility that a process like (44) may occur at any instant, if we were to do that then Wigner's theorem demands linearity. However, here one may perceive an outlet. The logical structure of quantum mechanics, as we are prepared to live with it, does not appear to preclude a trichotomy of evolutionary processes. One might just as well append nonlinear evolution as another special circumstance (a facetious standpoint); or one might discover that it governs the emergence of the special behaviour embodied in the rule (44), see Pearle[4]. The first option one can resolve by experiment[38], the second is interesting primarily from a philosophical viewpoint.

### C. A speculation on the process of measurement

The existence of our classical model certainly poses some very deep questions about the degree to which the orthodox probabilistic interpretation of linear quantum theory can be considered natural to the generalized dynamical framework that it inhabits. Whereas rule (23) proves useful to both linear quantum mechanics and our quantal model of classical mechanics, the rule (44) does not. One might argue that the success of our program was a mere accident, but the mathematical result that is responsible for it appears far from accidental[16].

Whereas the two theories of classical and quantal mechanics are generally considered to be wholly incompatible; we are now led to the view that the ultimate source of that incompatibility must lie within the projection postulate. Our view is that the probabilistic interpretation of  $\psi$  can never be explained using the deterministic linear evolution dictated

by the ordinary Schrödinger equation. If one desires an explanation, then we note that the rule (44) cannot survive as a possible instantaneous stochastic process within any nonlinear phase of evolution. Considering these facts, it seems that a resolution of the issue may creep by if one is prepared to consider the curious possibility that the classical regime could actually be a nonlinear regime of quantum theory.

Copenhagen orthodoxy characterises measurements as *special* processes involving metastable states of a macroscopic apparatus, which must always be described in classical terms[3, 49]. We have seen that exact classical mechanics requires a nonlinear quantum theory. Now suppose, for the sake of argument, that classical mechanics actually is an effective nonlinear form of quantum theory appropriate to macroscopic objects. Precisely how we are not sure, but the clue is that no physical system describable in classical terms could ever be considered *isolated* in the sense we understand that term to apply within linear quantum theory as defining the circumstances under which Wigner's theorem rules its decree. We already know that linear quantum theory is accurate for the dynamics of microscopic systems. Our primary supposition now demands some blend in between[46].

Let us now focus upon (44) as representing the major logical difficulty in our current understanding of quantum mechanics. The rule itself works so we accept it. Consider now its invocation. We use it primarily in those instances where the macroscopic world is coupled to the microscopic world in a manner such that the latter may induce a registerable and permanent change in the former[49]. Given our outlandish premise that classical mechanics may be an exact effective theory, it now appears self-evident that any process which couples a macroscopic device to a microscopic system would have to lie within the nonlinear regime of the presumed generalized theory. Since the only manner in which we are able to observe the microscopic regime is via such coupling, then it appears that we, meaning physicists and their instruments of observation, knowing that we are classical beings, must only ever have a nonlinear interaction with the subatomic world. When sufficiently isolated, the microscopic world may quite happily evolve in a linear fashion until such time as nonlinear creatures such as ourselves try and observe it. Since experience tells us that (44) applies in just these circumstances, then one is led to look for its motivation as an emergent property of quantal nonlinearity.

The idea is striking. It seems to us to be the only way one might ever resolve the difficulties offered by (44). It is plausible because (44) has a generic mathematical form (it

is always an inner product[38]). Were that not the case then there could be no universal trick that might accomplish such a feat. Such a trick appears necessary. However, granted the viewpoint, there is no obvious way to proceed. One would have to search for such a result in the belief that it does exist, when it may not. In this respect we are sad to admit that there appears to be very little hope for it[5].

### XIII. PHASE DYNAMICS AND ANHOLONOMY EFFECTS

Although we have lost operator dispersion as a signature of the quantal origins of this version of classical mechanics, it is thoroughly remarkable that in losing that quantal feature, classical mechanics actually gains for itself a new and quite surprising one. Within this version of classical mechanics there is now a natural  $U(1)$  quantal phase factor associated to any classical trajectory  $(q(t), p(t))$ . Classical mechanics acquires a natural *geometric phase*. It is here that the role of  $\hbar$  finally shows up. We shall demonstrate that the embedded quantal geometric phase, in combination with the dynamical phase, actually returns the *classical action* divided by  $\hbar$ .

The existence of a semiclassical interpretation of the Heisenberg–Weyl group phase factor has been noted before[18]. In this connection, it is already well-known, from semiclassical calculations, that Berry’s adiabatic quantal phase[25] enjoys an intimate relationship with Hannay’s adiabatic classical angle[40, 41]. Given that there is a good understanding of that topic, our discussion shall focus upon the classical analogue of Aharanov and Anandan’s[42] non-adiabatic version of Berry’s phase.

In our previous correction of the energy, we upset the phase dynamics in a most interesting way. A simple calculation, performed in appendix two, now reveals that the leading phase associated with any state  $\phi_0$  should be readjusted to read

$$|q(t), p(t); \phi_0\rangle = e^{i(\gamma(t) - \beta(t))} \hat{U}[q(t), p(t)] |\phi_0\rangle, \quad (45)$$

where the explicit formulæ for the phases are:

$$\gamma(t) = \frac{1}{\hbar} \int_0^t \left( \frac{\dot{q}p - \dot{p}q}{2} \right) d\tau, \quad (46)$$

$$\beta(t) = \frac{1}{\hbar} \int_0^t H_c(q, p) d\tau. \quad (47)$$

Observe that (45) contains the two usual contributions to the overall phase. This decomposition amounts to an application of the now standard technique for calculating the intrinsic phase anholonomy for a wavefunction circuit[35], as first discovered by Berry in his now famous paper[25]. The first phase  $\gamma(t)$  has a purely geometric origin, we recognise it as the Aharanov–Anandan form of Berry’s phase. The second phase  $\beta(t)$  is the usual dynamical phase.

As one might expect  $\gamma(t)$  has, in this one degree of freedom instance, a very simple geometrical interpretation as the integrated sectorial velocity (the rate of change of swept area in phase space, as measured from the origin, see Fig. 4). Significantly the area change has a sign attached which makes the swept area positive for rotation in a clockwise sense about the origin. On closed phase space circuits  $\Gamma$ , of arbitrary shape, we therefore expect a relation between the  $\gamma(\Gamma)$  accumulated on a loop and the area enclosed by that loop. Taking account of the implicit sign convention one finds that:

$$\int_0^T p \dot{q} dt = + \oint_{\Gamma} p dq \quad \text{and} \quad \int_0^T q \dot{p} dt = - \oint_{\Gamma} p dq,$$

where  $T$  is the time taken to execute one circuit. The negative sign arises in the second case because the sense of traversal is then that of a negative area contribution. The total result is

$$\gamma(\Gamma) = +1/\hbar \oint_{\Gamma} p dq. \quad (48)$$

So that one recognises the old Bohr–Sommerfeld quantization rule[49],

$$\oint_{\Gamma} p dq = 2\pi n \hbar,$$

in a new light as a constraint that  $\gamma(\Gamma)$  should be an integral multiple of  $2\pi$ .

The total phase  $\phi = \gamma - \beta$  can now be recast in the most interesting form:

$$\phi(\Gamma) = \frac{1}{\hbar} \oint_{\Gamma} p dq - \frac{1}{\hbar} \oint_{\Gamma} H(q, p) dt = \frac{1}{\hbar} \int_0^T L dt \quad (49)$$

which one immediately recognizes as the classical action. This relationship suggests that quantal geometric phases upon closed loops might well be interpreted as the natural action variables of quantum mechanics.

Note that the correspondence of  $\gamma(t)$  to the abbreviated classical action,  $S = \oint p dq$ , is confined to closed trajectories. Upon open trajectories the derivatives of both differ by a term of the form  $1/2d/dt(pq)$ . One can trace this phenomenon to an essential arbitrariness

connected with the manner in which one might choose to close any given open trajectory in phase space. We intend to return to deeper consideration of this question in a subsequent publication. At this stage it is instructive to conclude this section with two incidental comments.

Given our nonlinear quantization condition, one can now model any  $n$ -degree of freedom classical system using the creation and annihilation operators for  $n$ -degrees of freedom[18]. These have the standard commutation relations:

$$[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk} \quad j, k \in [1, n].$$

Under this extension, the Weyl operators generalize to

$$D(\alpha) \equiv \exp \left\{ \sum_{j=1}^n \alpha_j \hat{a}_j^\dagger - \alpha_j^* \hat{a}_j \right\},$$

with the obvious multiplication rule

$$D(\alpha)D(\beta) = \exp \left\{ \sum_{j=1}^n \text{Im}[\alpha_j \beta_j^*] \right\} D(\alpha + \beta).$$

Repeating an essentially identical argument one recovers the  $U(1)$  phase

$$\dot{\gamma}(t) = -\text{Im}[\dot{\alpha} \cdot \alpha^*].$$

This expression appears significant in respect of a throwaway remark made by Berry in his introductory article[44].

In the language of that paper the above equation amounts to the assignment of a semi-classical phase  $\gamma(\Gamma)$  on closed curves  $\Gamma$  bounding a surface  $S$ , where this is given by the flux of a classical 2-form,  $d\mathbf{p} \wedge d\mathbf{q}$ , as the simple expression

$$\gamma(\Gamma) = +\frac{1}{\hbar} \int \int_{\partial S=\Gamma} d\mathbf{p} \wedge d\mathbf{q}. \quad (50)$$

However, unlike the situation described in[44], this is not the angle averaged flux for an adiabatic excursion. In this situation we are dealing with the nonadiabatic phase. This means that the  $d$ 's now link  $\mathbf{p}$  and  $\mathbf{q}$ , so that the above relationship involves actual phase space forms rather than the parameter space forms of Berry's article[44]. This is interesting because it suggests the possibility that one might take a direct approach to the adiabatic phase as being an appropriately averaged swept area change in phase space due to the cyclic deformation of the phase space trajectories themselves.



Our second comment concerns the fact that the expression

$$\gamma(\Gamma) = -\text{Im} \oint_{\Gamma} \alpha^* d\alpha$$

is already a Poincaré integral invariant[20]. In terms of the classical variables  $(\mathbf{q}, \mathbf{p})$  it is the circuit integral

$$\gamma(\Gamma) = \frac{1}{\hbar} \oint_{\Gamma} \sum_{j=1}^n p_j dq_j,$$

which is equal to the area integral given previously.

This observation demonstrates that the quantal version of an integrable classical system, as we have defined it, will have  $n$  geometric phase “actions” associated with the  $n$  irreducible contours that exist upon any “torus” in the  $n$ -dimensional complex quantal phase space obtained via the generalized projection map  $\Pi[\psi(t)] = (\mathbf{q}(t), \mathbf{p}(t))$ . Considering the evolution of  $\psi(t)$  itself, we could say that it explores an  $n$ -dimensional sub-manifold of the infinite dimensional Hilbert space, which manifold might then be called a functional torus.

#### XIV. SOLUTIONS OF THE CLASSICAL EQUATION

We shall now exhibit a particular example of the classical Schrödinger equation in a familiar explicit representation. It is important to understand that this is just one example of what happens to be a perfectly general result.

For the one dimensional classical Hamiltonian,

$$H_c(q, p) = \frac{p^2}{2m} + V(q),$$

our corrected quantization prescription (26) gives the result

$$\hat{H}_q(\tilde{\psi}) = \frac{\langle \hat{p} \rangle^2}{2m} + V(\langle q \rangle) + \frac{\langle \hat{p} \rangle}{m}(\hat{p} - \langle \hat{p} \rangle) + \frac{\partial V}{\partial q}(\langle q \rangle)(\hat{q} - \langle \hat{q} \rangle). \quad (51)$$

This abstract operator Hamiltonian can now be carried into the standard Schrödinger representation via the usual rules:

$$\hat{q} \mapsto q, \quad \hat{p} \mapsto -i\hbar \frac{\partial}{\partial q}, \quad \text{and} \quad |\psi\rangle \mapsto \psi(q).$$

In the coordinate representation the required equation now corresponds to a nonlinear first-order integrodifferential equation of the explicit form:

$$i\hbar \left( \frac{\partial}{\partial t} + \frac{\langle \hat{p} \rangle}{m} \frac{\partial}{\partial q} \right) \psi(q, t) = \left( V(\langle \hat{q} \rangle) + \frac{\partial V}{\partial q}(\langle \hat{q} \rangle)(q - \langle \hat{q} \rangle) - \frac{\langle \hat{p} \rangle^2}{2m} \right) \psi(q, t), \quad (52)$$

where of course  $\langle \hat{q} \rangle$  and  $\langle \hat{p} \rangle$  must at all times satisfy the relations

$$\langle \hat{q} \rangle = \int_{-\infty}^{\infty} q \psi^*(q, t) \psi(q, t) dq \quad \text{and} \quad \langle \hat{p} \rangle = \int_{-\infty}^{\infty} -i\hbar \psi^*(q, t) \frac{\partial}{\partial q} \psi(q, t) dq.$$

The equation (52) is most unusual. At first sight the prospect of attempting to solve such a highly nonlinear equation might appear forbidding. However, here we know that the solutions of (52) are characterised by the fact that the position and momentum expectation values must follow precisely the classical trajectories.

The idea is then to seek solutions  $\psi(q, t)$  that are parametrised by the purely numerical functions of time

$$Q(t) \equiv \langle \psi(t) | \hat{q} | \psi(t) \rangle \quad \text{and} \quad P(t) \equiv \langle \psi(t) | \hat{p} | \psi(t) \rangle. \quad (53)$$

From our original Weyl operator considerations, we already know that the complete evolution operator solution of (52) must be

$$\tilde{U}[Q(t), P(t)] = e^{i\phi(t)} U[Q(t), P(t)],$$

where  $Q(t)$  and  $P(t)$  are parameters obtained via solving the purely classical equations of motion

$$\frac{dP}{dt} = -\frac{\partial V}{\partial q}(Q) \quad \text{and} \quad \frac{dQ}{dt} = +\frac{P}{m} \quad (54)$$

and the phase  $\phi(t)$  is fixed via the group anholonomy effects considered in the preceding section to be

$$\phi(t) = \frac{1}{\hbar} \int_0^t \left[ \frac{1}{2} \left( P(\tau) \dot{Q}(\tau) - Q(\tau) \dot{P}(\tau) \right) - \frac{P(\tau)^2}{2m} - V(Q(\tau)) \right] d\tau. \quad (55)$$

Using this insight we can now construct a general family of solutions to (52) and verify the condition (54) as an essential constraint upon the time-dependent parameters  $Q(t)$  and  $P(t)$ .

From §III C, equation (9), recall that  $U[Q(t), P(t)]$  has the convenient coordinate representation

$$U[Q(t), P(t)] = \exp \left\{ -\frac{iP(t)Q(t)}{2\hbar} \right\} \exp \left\{ +\frac{iP(t)q}{\hbar} \right\} \exp \left\{ -Q(t) \frac{\partial}{\partial q} \right\}.$$

Including the phase (55), we can now construct the required parametric family of solutions as the states

$$\psi(q, t) \equiv e^{i\phi(t)} \exp \left\{ -\frac{iP(t)Q(t)}{2\hbar} \right\} \exp \left\{ +\frac{iP(t)q}{\hbar} \right\} \psi_0(q - Q(t)), \quad (56)$$

where the time dependence of these enters purely via the group parameters  $Q(t)$  and  $P(t)$ . Provided that  $\phi_0(q)$  has both expectation values equal to zero, it follows that the constraint (53) is at all times trivially satisfied.

The idea is now to substitute the trial solution (56) into the wave equation (52) so as to discover a necessary condition upon the parameters  $Q(t)$  and  $P(t)$ . After some suppressed algebra, during which one must be careful to note that the special form of (56) enforces the equalities (53), one arrives at the required condition:

$$i\hbar \left( \frac{P(t)}{m} - \frac{dQ(t)}{dt} \right) \psi'_0(q - Q(t)) = \left( \frac{\partial V}{\partial q}(Q(t)) - \frac{dP(t)}{dt} \right) (q - Q(t)) \psi_0(q - Q(t)),$$

in which it should be noted that some phase factors have been conveniently cancelled. Examination of the above equation now shows that equality for all equivalence class representatives  $\psi_0(q)$  demands that  $Q(t)$  and  $P(t)$  should satisfy Hamilton's equations (54), as claimed.

So (52) actually admits an infinite family of physical travelling wave solutions. Moreover, the complete family of solutions must exhaust the set of all square integrable and appropriately differentiable representative functions  $\psi_0(q)$ . It is trivially assured that not one of these solutions exhibits dispersion phenomena of any kind, they simply follow the classical trajectories via the classical evolution of the functional parameters  $Q(t) = \langle \hat{q} \rangle$  and  $P(t) = \langle \hat{p} \rangle$ .

To consider a particular elementary example, one might set  $V(q) = 0$  so as to obtain the free particle equation

$$i\hbar \left( \frac{\partial}{\partial t} + \frac{\langle \hat{p} \rangle}{m} \frac{\partial}{\partial q} \right) \psi(q, t) = 0.$$

For the initial conditions  $Q(0) = Q_0$  and  $P(0) = P_0$  this equation is solved by the family

$$\psi(q, t) = \exp^{iP_0 q/\hbar} \phi_0(q - P_0 t/m - Q_0),$$

with  $\phi_0$  any state belonging to  $\tilde{\psi}(0, 0)$ .

## XV. RECOVERING STANDARD QUANTUM MECHANICS

We come now to the most remarkable result of all. Recall that the form of the corrected energy Hamiltonian (26):

$$\hat{H}_q(\tilde{\psi}) \equiv H_c + \frac{\partial H_c}{\partial p}(\hat{p} - \langle \hat{p} \rangle_\psi) + \frac{\partial H_c}{\partial q}(\hat{q} - \langle \hat{q} \rangle_\psi), \quad (57)$$

was constrained by the twin demands of a dynamical and energetic classical correspondence such that the position and momentum expectation values should track precisely the trajectories of any chosen classical Hamiltonian system.

In looking at the form of (57) one is naturally led to wonder about that theory which corresponds to the continuation of the implicit Taylor series to all orders. To investigate this possibility we require a way to write such an infinite operator valued series in compact form. As a simple way to explore that idea we elect to write the result in terms of a  $\psi$ -dependent *quantization operator*, which we define as follows[46]:

$$\hat{\mathcal{Q}}_\psi \equiv \exp \left\{ \sum_{k=1}^n (\hat{q}_k - \langle \hat{q}_k \rangle_\psi) \frac{\partial}{\partial q_k} + (\hat{p}_k - \langle \hat{p}_k \rangle_\psi) \frac{\partial}{\partial p_k} \right\}. \quad (58)$$

Under consideration is a classical system with  $n$  degrees of freedom and an associated quantal system having  $n$  canonically conjugate pairs of position and momentum operators  $\hat{q}_j$  and  $\hat{p}_j$ . The idea is that we should achieve the process of quantization via application of (58) to the classical function  $f_c(\mathbf{q}, \mathbf{p})$  so as to arrive at the quantal operator  $f_q(\hat{\mathbf{q}}, \hat{\mathbf{p}})$ .

Recognising that the ordinary  $c$ -number operational calculus permits us to write the identity

$$\exp\{(z - z_0) \frac{\partial}{\partial z} + (w - w_0) \frac{\partial}{\partial w}\} f(z_0, w_0) = f(z, w),$$

when it is understood that all partial derivatives are evaluated at the point  $(z_0, w_0)$ , we are led to define the action of (58) so that all  $c$ -number derivatives are evaluated in like manner at the quantal expectation values. Proceeding in this intuitive fashion one writes the expression

$$\hat{\mathcal{Q}}_\psi \circ f_c(\langle \hat{q}_1 \rangle_\psi, \dots, \langle \hat{q}_n \rangle_\psi; \langle \hat{p}_1 \rangle_\psi, \dots, \langle \hat{p}_n \rangle_\psi) = \hat{f}(\hat{q}_1, \dots, \hat{q}_n; \hat{p}_1, \dots, \hat{p}_n). \quad (59)$$

where it is expected that the formal properties of the implicit Taylor series shall ensure that the operator replacement is achieved in a manner that is independent of the choice made for the  $\psi$  expansion. We shall not prove this here, reserving that for a fuller treatment elsewhere.

For the present purpose it is clear that (58) is unique, and that it is a *linear* operator-valued map on a functional domain. Linearity of this mapping is here meant in the sense that

$$\hat{\mathcal{Q}}_\psi \circ [f + g] = \hat{\mathcal{Q}}_\psi \circ f + \hat{\mathcal{Q}}_\psi \circ g,$$

although it happens that the result is also a linear operator.

Given that we are to understand (58) in the ordinary sense of a differential operator power series expansion

$$\exp \mathcal{D} \equiv 1 + \mathcal{D} + 1/2! \mathcal{D}^2 \dots,$$

where the unusual feature of  $\mathcal{D}$  is simply the fact that it now carries within it some  $q$ -number coefficients, it is clear that the chosen mapping must achieve a replacement of the arguments of the classical Hamiltonian by canonically conjugate operators in a manner that automatically implements Weyl's symmetric ordering of the troublesome noncommuting factors, as generated by the cross derivatives of  $q_k$  and  $p_k$ . To cement the connection with canonical quantization, we remark that the  $\mathcal{Q}_\psi$  prescription can be shown to be equivalent to Weyl's operator Fourier transform[16]. As one might expect, given that equivalence, it is as well to mention also that one can use  $\mathcal{Q}_\psi$  to rederive Moyal's bracket[51] as the correct phase space analogue of the quantal commutator. Proofs shall be given in a forthcoming publication.

For the purpose of this article, the assertion is that  $\mathcal{Q}_\psi$  transforms any classical phase space function into a quantal operator equivalent, and that it does so in a manner which agrees with the standard canonical quantization procedure on cartesian coordinates, as augmented by the Weyl ordering rule. In this respect (58) has the important advantage that the procedure for operator replacement is now transparent.

### A. An example: the generalized harmonic oscillator

Since we here omit a full treatment of (58), a concrete example is useful to illustrate these points. Consider the ubiquitous one-dimensional generalized Harmonic oscillator:

$$H_c(q, p) = \frac{1}{2}(ap^2 + bpq + cq^2).$$

There are two possible quantized versions according to the chosen order of approximation. The nonlinear first-order version is

$$\begin{aligned} \hat{H}_q(\tilde{\psi}) = & 1/2(a\langle\hat{p}\rangle^2 + b\langle\hat{p}\rangle\langle\hat{q}\rangle + c\langle\hat{q}\rangle^2) \\ & + a\langle\hat{p}\rangle(\hat{p} - \langle\hat{p}\rangle) \\ & + b/2(\langle\hat{p}\rangle(\hat{q} - \langle\hat{q}\rangle) + \langle\hat{q}\rangle(\hat{p} - \langle\hat{p}\rangle)) \\ & + c\langle\hat{q}\rangle(\hat{q} - \langle\hat{q}\rangle) \end{aligned}$$

This truncation involves neglect of the second-order contribution

$$a/2(\hat{p} - \langle \hat{p} \rangle)^2 + b/2[(\hat{p} - \langle \hat{p} \rangle)(\hat{q} - \langle \hat{q} \rangle) + (\hat{p} - \langle \hat{p} \rangle)(\hat{q} - \langle \hat{q} \rangle)] + c/2(\hat{q} - \langle \hat{q} \rangle)^2.$$

Inclusion of the above term now recovers, in a natural fashion, the standard  $\psi$ -independent and correctly symmetrised result

$$\hat{H}_q = \frac{1}{2}(a\hat{p}^2 + b/2[\hat{p}\hat{q} + \hat{q}\hat{p}] + c\hat{q}^2).$$

In comparing the possible truncations of the operator valued Taylor series it is instructive to compute the energy expectation values. Doing that one discovers that the first-order version yields the correct classical energy:

$$E_c = \frac{1}{2}(a\langle \hat{p}^2 \rangle + b\langle \hat{p} \rangle \langle \hat{q} \rangle + c\langle \hat{q}^2 \rangle).$$

Upon inclusion of the second-order term, there appears a zero point contribution and one obtains the total result:

$$E_q = \frac{1}{2}(a\langle \hat{p}^2 \rangle + b/2\langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle + c\langle \hat{q}^2 \rangle).$$

As this example demonstrates, the classical Schrödinger equation can be viewed as representing a natural approximation to the full quantum theory. Any power series will do to represent the full quantal Hamiltonian. However, when considering the truncated series it is natural to adapt the approximation to the particular  $\psi$  in view. The successive truncations of a power series expanded about the expectation values of the instantaneous state could well be expected to provide a fair approximation to the correct quantum dynamics[47]. The two thoroughly remarkable features of this approach are that the first order approximation actually returns exact classical mechanics and that this approximation corresponds to a nonlinear dynamics.

## XVI. CONCLUSION

We began this paper by posing a simple question that induced us to construct a rather odd looking quantal dynamics based solely upon the desire to see classical behaviour mirrored in the evolution of entire classes of quantal wavefunctions. Remarkably, the resulting dynamical structure was fixed by this approach. Furthermore, the success of the procedure exposes the

crucial significance of the linear action of the Heisenberg–Weyl operator. The remarkable mathematical property of this group, namely that it provides a ray representation of the Abelian group of translations upon the plane, may be considered to be directly responsible for the success of our entire program. It also explains why classical mechanics acquires a phase factor. It seems that one need only adjoin the postulate that the fundamental kinematical group of motions upon the classical phase plane should have a projective structure and the entire result follows through of necessity[16].

Concerning the perceived connection with Weinberg’s theory[8, 9], it is interesting that both structures employ norm–preserving dynamics. In the case of Weinberg’s work there is an insistence upon the more restrictive property that the dynamics, and all observables, should remain unchanged under scaling of  $\psi$  by any non zero complex  $Z$ . This restriction to rays only can be incorporated into our structure by exploiting the norm–preserving property of our theory, so as to replace all appearances of  $\psi$  in the arguments of our nonlinear operators by  $\psi/n^{1/2}$ , where  $n$  is the preserved norm  $n = ||\psi||^2$ . An observation of this kind led us to discover a natural Weinberg analogue of our quantal model of classical mechanics[15]. We shall now describe the link between the present paper and that paper.

Considering the fact that our Hamiltonian (26) yields an energy observable with the classical value  $H_c(\langle\hat{q}\rangle, \langle\hat{p}\rangle)$  and noting that the central objects of Weinberg’s theory are observables  $h(\psi, \psi^*)$  that are homogeneous of degree one in both  $\psi$  and  $\psi^*$  we were led to attempt to carry our result directly into his theory. The obvious way to do this is to first generalize Weinberg’s  $\star$ –algebra of observables[8] to its natural infinite–dimensional analogue, and to then make the ansatz:

$$h(\psi, \psi^*) = n \cdot H_c(\langle\hat{q}\rangle, \langle\hat{p}\rangle),$$

where the homogeneity requirement is catered for by the revision

$$\langle\hat{q}\rangle_\psi = \langle\psi|\hat{q}|\psi\rangle/n \text{ and } \langle\hat{p}\rangle_\psi = \langle\psi|\hat{p}|\psi\rangle/n.$$

In the paper[15] we pursued this route as an entry point to Weinberg’s theory and uncovered a precisely analogous result. This situation provides very strong evidence for the conjecture that the two dynamical structures are ultimately equivalent. Given that the logical development of both approaches rests upon independent footing, one is inclined to view the territory under survey as being both rich and deep[52].

Certainly the success of our program is remarkable. In this respect, there are two distinct ways one might view the situation. On the one hand classical mechanics appears as a rather suprising *nonlinear approximation* to what is properly always an *exactly linear* theory. One might then understand this fact as providing a natural explanation of how it was that the great architects of linear quantum theory were able to find the correct structure out of simple analogies with the old theory of classical mechanics. On the other hand one might deem the non-dispersive nature of the solutions to the classical equation, the absence of superposition and the peculiarities of measurement theory as providing some minimal guidance that there truly is a nonlinear regime to quantum theory, and that this happens to lie in the domain of applicability of classical mechanics.

In conclusion, it has to be said that the author was more than a little shocked by the success of this naive program. So much so that we are prepared to entertain the possibility that the second bolder option may be correct. However unlikely the idea may seem, our preliminary investigations into an interpolative scheme of quantal nonlinearity[46] suggest that the possibility is alive, although it has to be said that the resulting family of nonlinear integrodifferential equations appears very odd. It therefore remains to be seen whether there is any direct physical role for nonlinear quantum mechanics.

Irrespective of the grave doubts that should rightly be attached to our presumption of a possible physical role for nonlinear quantum theory, it seems clear that our discovery of the classical Schrödinger equation must uncover a new conceptual standpoint that should be of inestimable value for the deeper elucidation of the connection between classical physics and quantum physics. For instance, the effective absence of  $\hbar$  from most features of our classical model provides an important window upon its general role in linear quantum theory. We shall take up this, and other related matters, in forthcoming publications.

## XVII. ACKNOWLEDGMENTS

I am grateful to both Professors B.H.J. McKellar and A.G. Klein for providing me with a means of financial support while this research was carried out. I further thank Prof. I.C. Percival, Prof. B.H.J. McKellar, Dr. N.E. Frankel, Dr. A.J. Davies, Dr. S. Kuyucak, Dr. V. Kowalenko and Dr. R. Volkas for some helpful discussion of various aspects of this result, for their general advice and for their invaluable encouragement.



Derivation of the Hamiltonian There is a very nice property of the Schrödinger equation, when cast in the evolution operator form

$$i\hbar \frac{d}{dt} \hat{U}(t, t_0) = \hat{H}(t) \hat{U}(t, t_0).$$

This concerns the fact that evolution operators are the natural analogues of phase space trajectories in the following special sense. Given any curve on the group manifold, namely an evolution operator trajectory  $\hat{U}(t, t_0)$ , we can deduce the quantal Hamiltonian appropriate to that trajectory.

Consider the group trajectory  $\hat{U}(t, t_0)$ . Applying only the essential unitarity constraint one can readily deduce the result

$$\hat{H}(t) \equiv i\hbar \left[ \frac{d}{dt} \hat{U}(t, t_0) \right] \hat{U}(t, t_0)^\dagger \quad (60)$$

as the defining equation for the Hamiltonian  $\hat{H}(t)$ .

Of interest here are group trajectories of the form

$$\tilde{D}(\alpha(t)) \equiv e^{i\phi(t)} D(\alpha(t))$$

where  $D(\alpha(t))$  is a member of the Heisenberg–Weyl group. The multiplication rule for this group is entirely sufficient to determine the necessary operator derivative, where this is defined as follows

$$\left[ \frac{d}{dt} \tilde{D}(\alpha(t)) \right] \tilde{D}(\alpha(t))^\dagger = \lim_{\delta t \rightarrow 0} \frac{\tilde{D}(\alpha(t) + \dot{\alpha}(t)\delta t) \tilde{D}(\alpha(t))^\dagger - \hat{I}}{\delta t}. \quad (61)$$

Using the fact that

$$\tilde{D}(\alpha(t))^\dagger = e^{-i\phi(t)} D(-\alpha(t))$$

one rewrites the above expression as

$$\left[ \frac{d}{dt} \tilde{D}(\alpha(t)) \right] \tilde{D}(\alpha(t))^\dagger = \lim_{\delta t \rightarrow 0} \frac{e^{+i(\dot{\phi} - \text{Im}[\dot{\alpha}\alpha^*])\delta t} D(\dot{\alpha}(t)\delta t) - \hat{I}}{\delta t}. \quad (62)$$

Expanding to first-order in  $\delta t$  then yields

$$\left[ \frac{d}{dt} \tilde{D}(\alpha(t)) \right] \tilde{D}(\alpha(t))^\dagger = +i(\dot{\phi} - \text{Im}[\dot{\alpha}\alpha^*])\hat{I} + \dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a} \quad (63)$$

as the final result. Note that the arbitrariness of  $\phi(t)$  shows up here as a contribution  $\dot{\phi}\hat{I}$  in the resulting Hamiltonian. One choice is especially convenient, that which cancels the term  $-\text{Im}[\dot{\alpha}\alpha^*]$ . To this end set

$$\phi(t) = + \int_0^t \text{Im}[\dot{\alpha}\alpha^*] dt$$

from which we obtain the Hamiltonian

$$-i\hat{H}(\alpha(t)) = \dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a}.$$

This choice of phase now yields the propagator equation of motion

$$\frac{d}{dt}\tilde{D}(\alpha(t)) = (\dot{\alpha}\hat{a}^\dagger - \dot{\alpha}^*\hat{a})\tilde{D}(\alpha(t)), \quad (64)$$

which verifies the result (13).

**Derivation of the Geometric Phase** To derive the geometric phase appropriate to any classical trajectory we shall work first with the  $D(\alpha)$  operators and then revert to classical phase space coordinates at the end of the argument.

Following the Aharanov–Anandan exposition[42] of the geometric phase as being associated with a path in the projective Hilbert space, and noting that our propagators are always Weyl operators, one can consider an arbitrary initial state  $\psi(0)$ , and a propagator  $D(\alpha(t))$ , as determining the ray space trajectory:

$$|\psi(t)\rangle = D(\alpha(t))|\psi(0)\rangle.$$

We can then calculate the geometric phase using the rule[42]

$$\dot{\gamma}(t) = i\langle\psi(0)|D(\alpha(t))^\dagger[\frac{d}{dt}D(\alpha(t))]| \psi(0)\rangle. \quad (65)$$

This way of generating an arbitrary trajectory builds up increments in the phase  $\gamma(t)$  via the geodesic transport of  $|\psi(0)\rangle$ , out along a straight line from the origin, along a path segment dictated by that upon the group, and thence back to the origin[43].

Using essentially the same argument as in appendix one, the time derivative of the phase is now easily calculated to be

$$\dot{\gamma} = -\text{Im}[\dot{\alpha}\alpha^*],$$

whence we obtain the desired result

$$\gamma(t) = -\int_0^t \text{Im}[\dot{\alpha}\alpha^*] d\tau. \quad (66)$$

It is instructive to use the substitution  $\alpha = c_q q + i c_p p$  and  $c_q c_p = 1/2\hbar$  so as to rewrite this phase as

$$\gamma(t) = \int_0^t \frac{p\dot{q} - \dot{p}q}{2\hbar} d\tau. \quad (67)$$

This assigns a natural semiclassical Aharanov–Anandan quantal geometric phase to an arbitrary classical trajectory. Note that this phase is a geometric property of the trajectory alone and the chosen criterion for closing open trajectories. In this respect, other closing criteria are possible, such as closing to the  $q$  or  $p$  axes, rather than to the origin. These shall of course yield different values for the integrated phase upon open trajectories. Moreover, the manner of closing is not preserved under canonical transformations. However, upon naturally closed trajectories the arbitrariness is removed and the geometric phase, being a well defined phase space area, must be invariant under all classical canonical transformations. The only phase freedom is then that of adding a multiple of the unit operator to the Hamiltonian. This can be interpreted as an expression of the standard freedom to change the energy origin in both classical and quantum mechanics.

To compute the overall phase, we must now subtract the dynamical contribution

$$\beta(t) = \frac{1}{\hbar} \int_0^t \langle \psi | \hat{H}_q(\tilde{\psi}) | \psi \rangle d\tau,$$

which, given the rule (26), is easily seen to be

$$\beta(t) = \frac{1}{\hbar} \int_0^t H_c(q, p) d\tau.$$

In this manner one recovers equations (45), (46) and (47), as required.

- 
- [1] L. de Broglie, “nonlinear Wave Mechanics—A Causal Interpretation,” Elsevier, Amsterdam, 1950.
  - [2] I. Biàlynicki-Birula and J. Mycielski, *Ann. Phys. (N.Y.)* **100** (1976), 62; and references therein.
  - [3] For a collection of seminal papers addressing this question see: “Quantum Theory and Measurement,” (J.A. Wheeler and W.H. Zurek, Eds.), Princeton Univ. Press, Princeton, 1983.
  - [4] P. Pearle, *Phys Rev D* **13** (1976), 857.
  - [5] Bell’s theorem presents the most significant obstruction to such a program, for his articles on this topic see: J.S. Bell, “Speakable and Unspeakable in Quantum Mechanics,” Cambridge Univ Press, London, 1988. In this connection, see also the recent paper: J. Polchinski, *Phys. Rev. Lett.* **66** (1991), 397.
  - [6] C.G. Shull, D.K. Attwood, J. Arthur and M. Horne, *Phys. Rev. Lett.* **44** (1980), 765.

- [7] R. Gähler, A.G. Klein and A. Zeilinger, *Phys. Rev.* **23** (1981), 1611.
- [8] S. Weinberg, *Ann. Phys. (N.Y.)* **194** (1989), 336; and references therein.
- [9] S. Weinberg, *Phys. Rev. Lett.* **62** (1989), 485.
- [10] F. Strocchi, *Rev. Mod. Phys.* **38** (1966), 36.
- [11] A. Heslot, *Phys. Rev. D* **31** (1985), 1341; and references therein.
- [12] See, for example, the recent book: M.C. Gutzwiller, “Chaos in Classical and Quantum Physics,” Springer-Verlag, New York, 1990; and references therein.
- [13] J. Ford, G. Mantica and G.H. Ristow, *Physica* **D50**, 493 (1991)
- [14] Numerous high precision tests of linearity have been performed using the Weinberg theory (see Ref. 9. for the proposal). Early examples include: J.J. Bollinger, D.J. Heinzen, W.M. Itano, S.L. Gilbert and D.J. Wineland, *Phys. Rev. Lett.* **63** (1989), 1031; T.E. Chupp and R.J. Hoare, *Phys. Rev. Lett.* **64** (1990), 2261; R.L. Walsworth, I.F. Silvera, E.M. Mattison and R.F.C. Vessot, *Phys. Rev. Lett.* **64** (1990), 2599; For example, Bollinger et al bounded the fraction of binding energy per nucleon of the  $^9\text{Be}^+$  nucleus that could be due to nonlinear corrections at less than  $4 \times 10^{-27}$ . These tests certainly rule out nonlinearity for *isolated* microscopic systems and so verify the logic of Wigner’s theorem.
- [15] K.R.W. Jones, University of Melbourne, Preprint No. UM-P-91/90 (1991).
- [16] H. Weyl, “The Theory of Groups and Quantum Mechanics,” pp272–280, Dover,

- New York, 1950; it should be pointed out that the treatment presented in these few pages forms the mathematical backbone of our result. In particular, Hermann Weyl's proof that there is only one irreducible ray representation of a 2-parameter continuous Abelian group (pp277–280) is central to enforcing the uniqueness of our procedure. It follows from this fact that there is only one route via which classical mechanics may acquire a phase,  $\psi$  and  $\hbar$ .
- [17] J.R. Klauder and B.-S. Skagerstam, “Coherent States: Applications in Physics and Mathematical Physics,” World Scientific, Singapore, 1985.
  - [18] A. Perelemov, “Generalized Coherent States and Their Applications,” Springer–Verlag, Berlin, 1986.
  - [19] W.-M. Zhang, D.-H. Feng and R. Gilmore, *Rev. Mod. Phys.* **62** (1990), 867.
  - [20] V.I. Arnold, “Mathematical Methods of Classical Mechanics,” 2nd ed., Springer–Verlag, Berlin, 1989.
  - [21] G.W. Mackey, “The Mathematical Foundations of Quantum Mechanics,” Benjamin, New York, 1963.
  - [22] M.H. Stone, *Proc. Nat. Acad. Sci.* **16** (1932), 172.
  - [23] J. von Neumann, *Math. Ann.* **104** (1931), 570.
  - [24] E. Prugovečki, “Quantum Mechanics in Hilbert Space,” pp329–347, Academic Press, New York, 1971.
  - [25] M.V. Berry, *Proc. Roy. Soc. Lond.* **A392** (1984), 45.
  - [26] P.A.M. Dirac, *Fields and Quanta* **3** (1972) 139.
  - [27] J.R. Klauder, *J. Math. Phys.* **4** (1963), 1058.
  - [28] A fully elaborated viewpoint of this kind is presented in: J.R. Klauder, *J. Math. Phys* **4** (1963), 1055
  - [29] For an elementary treatment of Banach space methods see: W. Rudin, “Real and Complex Analysis,” Chap. 5, 3rd ed., McGraw–Hill, New York, 1987.
  - [30] Ref. 17 pp20–25.
  - [31] There is no preferred origin in this space. The transformations  $\hat{p}' \equiv U[q, p]^\dagger \hat{p} U[q, p]$  and  $\hat{q}' \equiv U[q, p]^\dagger \hat{q} U[q, p]$  preserve the commutation relations, and thus the group algebra. Under this change the old origin at the unprimed class  $\tilde{\psi}(0, 0)$  shifts to the new origin at the primed class  $\tilde{\psi}'(0, 0)$ , whose members are identical up to phase with those of the unprimed class  $\tilde{\psi}(-q, -p)$ .
  - [32] A deep underlying reason for this supposed connection may be perceived within the group

- identity:  $U(n) = O(2n) \cap Sp(2n)$ , (see Ref. 20 p 225). This identity demonstrates that the finite-dimensional version of the generalized dynamical system in view, be it in Weinberg form or our own, is a specialized type of classical mechanics where the fundamental classical symmetry group  $Sp(2n)$ , to which all “infinitesimal generators” must belong, is further constrained by the norm-preservation property to joint membership of  $O(2n)$ . The result is then  $U(n)$  the fundamental symmetry group of quantum mechanics. This observation shows the generalized theory to be a *very natural* mathematical structure.
- [33] For a recent review of this argument see: T.F. Jordan, *Am. J. Phys.* **59** (1991), 606; and references therein.
  - [34] H. Everett, *PhD Thesis*, in “The Many-Worlds Interpretation of Quantum Mechanics,” B. de Witt and N. Graham Eds., Princeton, New Jersey, 1973.
  - [35] See the reprint volume: “Geometric Phases in Physics,” (A. Shapere and F. Wilczek Eds.), World Scientific, Singapore, 1989.
  - [36] K. Gottfried, “Quantum Mechanics Vol. I: Fundamentals,” p68, Benjamin, New York, 1966.
  - [37] K.G. Kay, *Phys. Rev. A* **42** (1990), 3718.
  - [38] A direct empirical test of quantal linearity appears possible once one has a means of measuring quantum states. The idea is that one should look for a violation of the inner product preserving property of ordinary linear quantum theory. An approach to quantum measurement theory appropriate to this task appears in: K.R.W. Jones, *Ann. Phys. (N.Y.)* **207** (1991), 140; K.R.W. Jones, *J. Phys. A* **24** (1991), 121.
  - [39] R. Penrose, in “300 Years of Gravitation,” pp25–34, S. Hawking and W. Israel Eds., Cambridge University Press, London, 1989.
  - [40] J.H. Hannay, *J. Phys. A: Math. Gen.* **18** (1985), 221.
  - [41] M.V. Berry, *J. Phys. A: Math. Gen.* **18** (1985), 15.
  - [42] Y. Aharonov and J. Anandan, *Phys. Rev. Lett.* **58** (1987), 1593.
  - [43] This method of calculation represents an elementary application of the approach outlined in: J. Samuel and R. Bhandari, *Phys. Rev. Lett.* **60** (1988), 2339.
  - [44] Ref. 35 pp 7–28.
  - [45] P.A.M. Dirac, “The Principles of Quantum Mechanics,” 4th ed., Oxford Univ. Press, London, 1958.
  - [46] The first example of a model that interpolates between the classical and quantal regimes of

- $\psi$ -dynamics appears in the unpublished manuscript: K.R.W. Jones, University of Melbourne Preprint No. UM-P-91/47 (1991). There the reduction of linear quantum mechanics to classical mechanics occurs as a single dimensionless classicality parameter  $\lambda$  is taken from the value  $\lambda = 1$  to the value  $\lambda = 0$ , while  $\hbar$  is kept fixed.
- [47] For a novel proof of the standard classical limit using an approach of this kind see: A. Messiah, “Quantum Mechanics Vol I,” p216, North Holland, Amsterdam, 1961; However, there the connection with nonlinearity is not recognised. The same comment goes for Ref. 37.
  - [48] Y.S. Kim and E.P. Wigner, *Am. J. Phys.* **58** (1990), 439; and references therein.
  - [49] D. Bohm, “Quantum Theory,” Dover, New York, 1951.
  - [50] H.J. Groenwold, *Physica* **12** (1946), 405; L. van Hove, *Mem. Acad. Roy. Belg.* **26** (1951), 61; for a proof of this result see V. Guillemin and S. Sternberg, “Symplectic Techniques in Physics,” pp101-104, Cambridge University Press, London 1984.
  - [51] J.E. Moyal, *Proc. Camb. Phil. Soc.* **45** (1949), 99.
  - [52] If the structure of nonlinear quantum mechanics is not used somewhere (*where* appears to be the operative question) then, in the light of the remark at Ref. 32, Nature simply missed out upon using an exceedingly elegant mathematical structure.

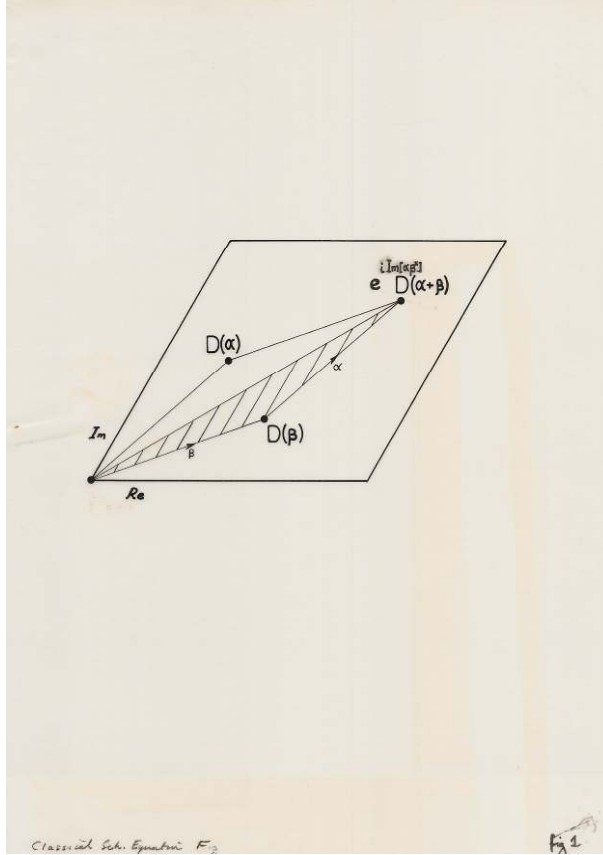


FIG. 1: The parameter space of Weyl operators is a plane, multiplication of two of them, say  $D(\alpha)$  and  $D(\beta)$ , leads to the new unitary operator  $e^{i\text{Im}[\alpha\beta^*]}D(\alpha+\beta)$ , where the phase factor derives from the ray character of this representation of the Abelian group of translations upon the plane..



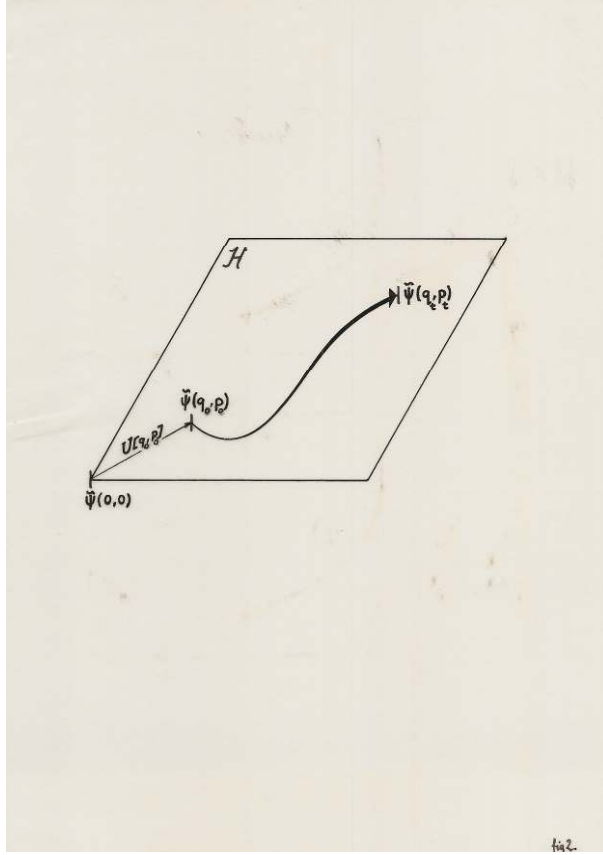


FIG. 2: Hilbert space  $\mathcal{H}$  considered as the Weyl translates of a special class of states  $\tilde{\psi}(0,0) = \{\phi_0\}$ , where  $\phi_0$  runs through all states having both expectation values equal to zero. The points  $\tilde{\psi}(q,p)$  in this special quantal phase space thus represent entire classes of wavefunctions which enjoy a one-one correspondence with the phase space points of ordinary classical mechanics.

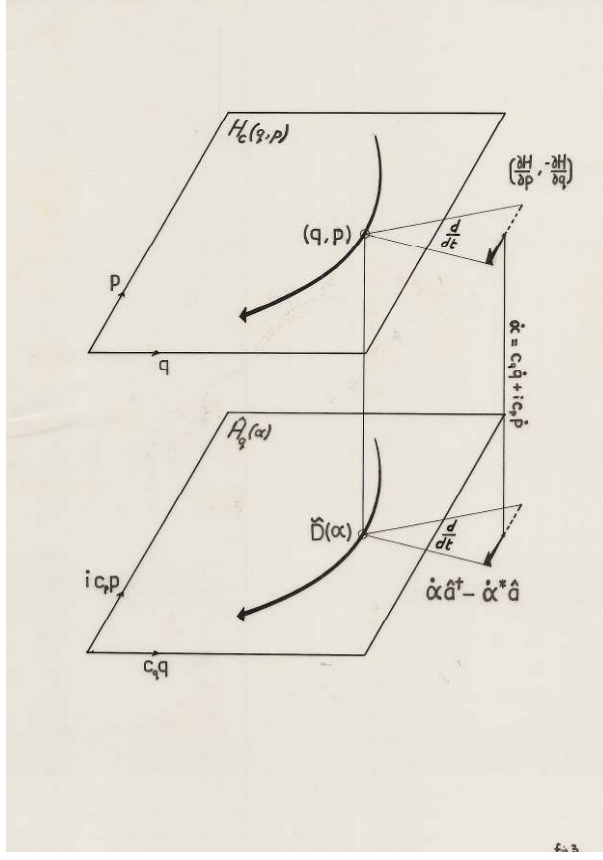


FIG. 3: The classical and quantal phase planes. In the upper classical plane there is a trajectory  $(q(t), p(t))$  being a solution to the Hamiltonian system  $H_c(q, p)$ . In the lower quantal plane there is a Weyl operator trajectory  $\tilde{D}(\alpha(t))$ , which we fix up to an arbitrary time-dependent phase via the projection  $c_q q(t) + i c_p p(t) \mapsto \alpha(t)$ . The choice of  $c_q$  and  $c_p$  sets the scale relation between both planes, where  $\hbar = [2c_q c_p]^{-1}$ .

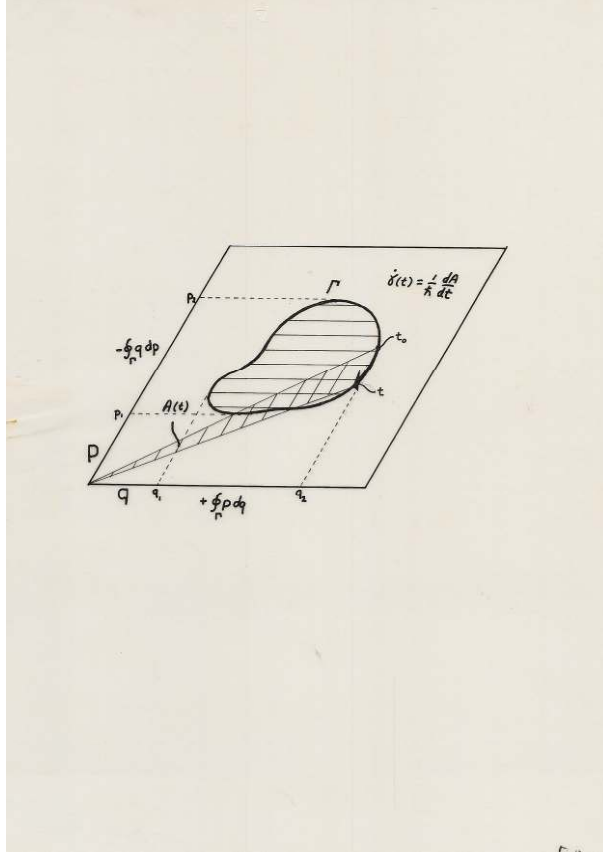


FIG. 4: The classical geometric phase embedded in ordinary classical phase space. The form of  $\gamma(t)$  is such that it equals the integrated sectorial velocity, as measured from the origin. Upon closed circuits  $\Gamma$ , the phase  $\gamma(\Gamma)$  can be decomposed into the two terms:  $+1/2\hbar \oint_{\Gamma} p dq$  and  $-1/2\hbar \oint_{\Gamma} q dp$ . Accounting for the sense of traversal, these terms are seen to be numerically equal. Hence  $\gamma(\Gamma) = +1/\hbar \oint_{\Gamma} p dq$ .